MATH 505a Fall 2017 Qual Solution Attempts

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August 4, 2022

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Problem 1

Let X be uniform on [1,5], let Y be uniform on [0,1], and assume that X and Y are independent.

(a)

Compute the probability density function of the product XY.

Solution.

$$\begin{aligned}\mathbb{P}(XY \le t) &= \mathbb{P}(Y \le \frac{t}{X}) \\ &= \int \mathbb{P}(Y \le \frac{t}{x}) f_X(x)dx\end{aligned}$$

Case 1. $0 < t \le 1$,

$$\begin{aligned}\mathbb{P}(Y \le \frac{t}{X}) &= \int_1^5 \frac{t}{x}\frac{1}{4}dx \\ &= \frac{t}{4}\ln(5)\end{aligned}$$

Case 2. $1 < t \le 5$,

$$\begin{aligned}\mathbb{P}(Y \le \frac{t}{X}) &= \int_1^t 1 \cdot \frac{1}{4}dx + \int_t^5 \frac{t}{x} \cdot \frac{1}{4}dx \\ &= \frac{1}{4}(t-1) + \frac{t}{4}\ln(\frac{5}{t})\end{aligned}$$

Therefore, by differentiating:

$$f_{XY}(t) = \begin{cases} \frac{\ln(5)}{4}, & t \in (0,1], \\ \frac{1}{4}\ln\left(\frac{5}{t}\right), & t \in (1,5], \\ 0, & \text{otherwise.} \end{cases}$$

(b)

Compute the cumulative distribution function of the ratio X/Y.

Solution.

$$\begin{aligned}\mathbb{P}(X/Y \le t) &= \mathbb{P}(Y \ge X/t)\\ &= \int \mathbb{P}(Y \ge x/t) f_X(x)dx\end{aligned}$$

Case 1. $1 < t \le 5$,

$$\begin{aligned}\mathbb{P}(Y \ge X/t) &= \int_1^t (1 - \frac{x}{t}) \cdot \frac{1}{4}dx\\ &= \frac{t}{8} + \frac{1}{8t} - \frac{1}{4}\end{aligned}$$

Case 2. $t \ge 5$,

$$\begin{aligned}\mathbb{P}(Y \ge X/t) &= \int_1^5 (1 - \frac{x}{t}) \cdot \frac{1}{4}dx\\ &= 1 - \frac{3}{t}\end{aligned}$$

(c)

Compute the characteristic function of the sum $X + Y$.

Solution.

$$\begin{aligned}\mathbb{E}(e^{it(X+Y)}) &= \mathbb{E}(e^{itX})\mathbb{E}(e^{itY})\\ &= \int_1^5 e^{itx}\frac{1}{4}dx \int_0^1 e^{ity}dy\\ &= -\frac{1}{4t^2}(e^{5it} - e^{it})(e^{it} - 1)\end{aligned}$$

(d)

Compute the moment generating function of the random variable $X - \ln(Y)$.

Solution.

$$\begin{aligned}\mathbb{E}(e^{t(X-\ln Y)}) &= \mathbb{E}(e^{tX} \cdot e^{-t\ln Y})\\ &= \mathbb{E}(e^{tX}) \cdot \mathbb{E}(Y^{-t})\\ &= \int_1^5 e^{tx}\frac{1}{4}dx \int_0^1 y^{-t}dy\end{aligned}$$

Notice that the right multiplicand's integrability depends on t, so

$$\mathbb{E}(e^{t(X-\ln Y)}) = \begin{cases} \frac{e^{5t}-e^{t}}{4t(1-t)}, & t < 1, \\ \infty, & t \geq 1 \end{cases}$$

Problem 2

An urn contains $2n$ balls, coming in pairs: two balls are labeled "1", two balls are labeled "2",..., two balls are labeled "n". A sample of size n is taken without replacement. Denote by N the number of pairs in the sample. Compute the expected value and the variance of N. **You do not need to simplify the expression for the variance.**

Solution. Let X_i be the indicator function of the pair of balls labeled "i" are selected. And the probability of any pair being selected is the ratio of the number of combinations to select $n-2$ balls from the rest of $2n-2$ balls and the total number of combinations to select n balls from $2n$ balls.

$$\begin{aligned}
\mathbb{E}(N) &= \mathbb{E}(\sum_{i=1}^{n} X_i) \\
&= \sum_{i=1}^{n} \mathbb{P}(X_i = 1) \\
&= n \cdot \frac{\binom{2n-2}{n-2}}{\binom{2n}{n}} \\
&= \frac{n(n-1)}{2(2n-1)} \\
\mathbb{E}(X_i^2) &= \mathbb{E}(X_i) \\
&= \frac{n-1}{2(2n-1)}
\end{aligned}$$

Notice that the probability of two pairs being selected is the ratio of the number of combinations to select $n-4$ balls from the rest of $2n-4$ balls and the total number of combinations to select n balls from $2n$ balls. so for $i \neq j$,

$$\begin{aligned}
\mathbb{E}(X_i X_j) &= \mathbb{P}(X_i = 1,\ X_j = 1) \\
&= \frac{\binom{2n-4}{n-4}}{\binom{2n}{n}} \\
&= \frac{n(n-1)(n-2)(n-3)}{2n(2n-1)(2n-2)(2n-3)}
\end{aligned}$$

$$\begin{aligned}
Var(N) &= \mathbb{E}(N^2) - \mathbb{E}(N)^2 \\
&= \sum_{i=1}^{n} \mathbb{E}(X_i^2) + \sum_{i \neq j}^{n} \mathbb{E}(X_i X_j) - \mathbb{E}(N)^2 \\
&= n \cdot \frac{(n-1)}{2(2n-1)} + n(n-1) \cdot \frac{n(n-1)(n-2)(n-3)}{2n(2n-1)(2n-2)(2n-3)} - \left(\frac{n(n-1)}{2(2n-1)}\right)^2
\end{aligned}$$

Problem 3

Let $U_1, U_2, ...$ be iid random variables, uniformly distributed on [0,1], and let N be a Poisson random variable with mean value equal to one. Assume that N is independent of $U_1, U_2, ...$ and define

$$Y = \begin{cases} 0, & \text{if } N = 0, \\ \max_{1\le i\le N} U_i, & \text{if } N > 0. \end{cases}$$

Compute the expected value of Y.

Solution. First we compute the expectation of $max_{1\le i\le k} U_i$ for some $k \ge 1$. For $0 < t < 1$,

$$\begin{aligned} \mathbb{P}(\max_{1\le i\le k} U_i \le t) &= \prod_{i=1}^{k} \mathbb{P}(U_i \le t) \\ &= t^k \end{aligned}$$

Since U_i only takes nonnegative value,

$$\begin{aligned} \mathbb{E}(\max_{1\le i\le k} U_i) &= \int_0^1 (1 - t^k) dt \\ &= \frac{k}{k+1} \end{aligned}$$

We compute the expectation of Y by conditioning on N,

$$\begin{aligned} \mathbb{E}(Y) &= \sum_{k=0}^{\infty} \mathbb{E}(Y|N = k)\mathbb{P}(N = k) \\ &= 0 + \sum_{k=1}^{\infty} \frac{k}{k+1} e^{-1} \frac{1}{k!} \\ &= e^{-1} \sum_{k=1}^{\infty} \left(\frac{1}{k!} - \frac{1}{(k+1)!} \right) \\ &= e^{-1} \end{aligned}$$