# MATH 505a Spring 2016 Qual Solution Attempts

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## Problem 1

A stick of length 1 is broken at a point uniformly distributed over its length.

### (a)

Find the mean and variance of the sum $S$ of the squares of the lengths of the two pieces.

*Solution.* Let $U$ be the location of the breaking point.

$$\begin{aligned}
\mathbb{E}(S) &= \mathbb{E}(U^2 + (1-U)^2) \\
&= \mathbb{E}(2U^2 - 2U + 1) \\
&= 2 \cdot \frac{1}{3} - 2 \cdot \frac{1}{2} + 1 \\
&= \frac{2}{3}
\end{aligned}$$

$$\begin{aligned}
\operatorname{Var}(S) &= \mathbb{E}(S^2) - (\mathbb{E}(S))^2 \\
&= \mathbb{E}((2U^2 - 2U + 1)^2) - \left(\frac{2}{3}\right)^2 \\
&= \mathbb{E}(4U^4 - 8U^3 + 8U^2 - 4U + 1) - \frac{4}{9} \\
&= 4 \cdot \frac{1}{5} - 8 \cdot \frac{1}{4} + 8 \cdot \frac{1}{3} - 4 \cdot \frac{1}{2} + 1 - \frac{4}{9} \\
&= \frac{1}{45}
\end{aligned}$$

### (b)

Find the density function of the product $M$ of the lengths of the two pieces. Note that $M \in [0, \frac{1}{4}]$.

*Solution.*

$$\begin{aligned}\mathbb{P}(M \le t) &= \mathbb{P}(U(1-U) \le t)\\ &= \mathbb{P}(U \le \frac{1}{2}(1-\sqrt{1-4t})) + \mathbb{P}(U \ge \frac{1}{2}(1+\sqrt{1-4t}))\\ &= 1-\sqrt{1-4t},\ 0 \le t \le \frac{1}{4}\\ f_M(t) &= \frac{d}{dt}(1-\sqrt{1-4t})\\ &= \frac{2}{\sqrt{1-4t}},\ 0 \le t \le \frac{1}{4}\end{aligned}$$

# Problem 2

There are two types of batteries in a bin. The life span of type $i$ is an exponential random variable with mean $\mu_i, i = 1, 2$. The probability of type $i$ battery to be chosen is $p_i$, with $p_1 + p_2 = 1$. Suppose a randomly chosen battery is still operating after $t$ hours. What is the probability that it will still be operating after an additional $s$ hours?

*Solution.* Denote the life span for type 1 and 2 battery as $B_1, B_2$. Let $B$ be the life span of the chosen battery.

$$\begin{aligned}\mathbb{P}(B > s+t | B > t) &= \mathbb{P}(B_1 > s+t | B_1 > t)p_1 + \mathbb{P}(B_2 > s+t | B_2 > t)p_2\\ &\overset{(*)}{=} \mathbb{P}(B_1 > s)p_1 + \mathbb{P}(B_2 > s)p_2\\ &= p_1 e^{-\mu_1 s} + p_2 e^{-\mu_2 s}\end{aligned}$$

$(*)$ Exponential random variables are memory-less. Proof is omitted.

# Problem 3

Fix positive integers $m \le n$ with $n > 4$. Suppose $m$ people sit at a circular table with $n$ seats, with all $\binom{n}{m}$ seating equally likely. A seat is called *isolated* if it is occupied and both adjacent seats are vacant. Find the mean and variance of the number of isolated seats.

*Solution.* Let $X_i$ be the indicator function of $i$th seat being isolated. Let $N = \sum_{i=1}^{n}$ be the total number of isolated seats. Clearly when $n < m+2$, $N = 0$, $\mathbb{E}(N) = 0$, $\mathrm{Var}(N) = 0$. Assume $n \ge m+2$,

$$\begin{aligned}\mathbb{E}(N) &= \sum_{i=1}^{n} \mathbb{E}(X_i)\\ &= \sum_{i=1}^{n} \binom{n-3}{m-1} \Big/ \binom{n}{m}\\ &= n \cdot \binom{n-3}{m-1} \Big/ \binom{n}{m}\end{aligned}$$

$$\mathbb{E}(X_i^2) = \mathbb{E}(X_i) = \binom{n-3}{m-1} \Big/ \binom{n}{m}$$

For $i \neq j$, we want to compute the probability when both $i$th and $j$th seats are isolated.

*Case 1.* When $|i-j| = 1$ or $|i-j| = n-1$(since the end is connect to the start), it's impossible since they are next to each other and both being occupied, so

$$\mathbb{E}(X_iX_j) = 0$$

*Case 2.* When $|i-j| = 2$ or $|i-j| = n-2$, we need at least 3 vacant seats otherwise it's impossible, so

$$\mathbb{E}(X_iX_j) = \begin{cases} \binom{n-5}{m-2} \Big/ \binom{n}{m} & n \geq m+3 \\ 0 & o.w. \end{cases}$$

*Case 3.* When $2 < |i-j| < n-2$, we need at least 4 vacant seats otherwise it's impossible, so

$$\mathbb{E}(X_iX_j) = \begin{cases} \binom{n-6}{m-2} \Big/ \binom{n}{m} & n \geq m+4 \\ 0 & o.w. \end{cases}$$

Now, we compute the variance for different range of $n-m$:

When $n - m \geq 4$,

$$\begin{aligned}
\operatorname{Var}(N) &= \mathbb{E}(N^2) - (\mathbb{E}(N))^2 \\
&= \sum_{i=1}^{n} \mathbb{E}(X_i^2) + \sum_{i \neq j}^{n} \mathbb{E}(X_iX_j) - (\mathbb{E}(N))^2 \\
&= \mathbb{E}(N) - (\mathbb{E}(N))^2 + \sum_{\substack{|i-j|=2 \\ \text{or } n-2}} \binom{n-5}{m-2} \Big/ \binom{n}{m} + \sum_{\substack{2<|i-j| \\ |i-j|<n-2}} \binom{n-6}{m-2} \Big/ \binom{n}{m} \\
&= \mathbb{E}(N) - (\mathbb{E}(N))^2 + 2n \cdot \binom{n-5}{m-2} \Big/ \binom{n}{m} + (n(n-1) - 4n) \cdot \binom{n-6}{m-2} \Big/ \binom{n}{m} \\
&= \binom{n-3}{m-1} \Big/ \binom{n}{m} - \left[ \binom{n-3}{m-1} \Big/ \binom{n}{m} \right]^2 + 2n \cdot \binom{n-5}{m-2} \Big/ \binom{n}{m} + (n(n-1) - 4n) \cdot \binom{n-6}{m-2} \Big/ \binom{n}{m}
\end{aligned}$$

When $n - m = 3$, *Case 3* is impossible, so

$$\begin{aligned}
\operatorname{Var}(N) &= \mathbb{E}(N) - (\mathbb{E}(N))^2 + \sum_{i \neq j}^{n} \mathbb{E}(X_iX_j) \\
&= \binom{n-3}{m-1} \Big/ \binom{n}{m} - \left[ \binom{n-3}{m-1} \Big/ \binom{n}{m} \right]^2 + \sum_{\substack{|i-j|=2 \\ \text{or } n-2}} \binom{n-5}{m-2} \Big/ \binom{n}{m}
\end{aligned}$$

When $n - m = 2$, it is impossible to have more than one isolated seats, which means we won't have any nonzero $\mathbb{E}(X_iX_j)$, $i \neq j$

$$\begin{aligned}
\operatorname{Var}(N) &= \mathbb{E}(N) - (\mathbb{E}(N))^2 \\
&= \binom{n-3}{m-1} \Big/ \binom{n}{m} - \left[ \binom{n-3}{m-1} \Big/ \binom{n}{m} \right]^2
\end{aligned}$$