MATH 505a QUALIFYING EXAM Monday, February 9, 2015. One hour and 50 minutes, starting at 5pm.

Ideas for solutions.

1. *Let $X_n,\ n \geq 1$, be independent random variables such that each X_n has Poisson distribution with mean λ_n. Prove that if $\sum_{n\geq 1} \lambda_n = +\infty$, then*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} X_k}{\sum_{k=1}^{n} \lambda_k} = 1$$

in probability.

Solution. Direct by Chebyshev inequality

$$\mathbb{P}(|Y - \mathbb{E}Y| > a) \leq \frac{\mathrm{Var}(Y)}{a^2}$$

with $Y = \sum X_k,\ \mathbb{E}Y = \sum \lambda_k = \mathrm{Var}(Y),\ a = \varepsilon \mathbb{E}Y$.

2. *A deck of cards is shuffled thoroughly. Someone goes through all 52 cards, scoring 1 each time 2 cards of the same value are consecutive. For example 9H,8H,7D,6C,7S,7H,7C, scores 2, once due to 7 of spades next to 7 of hearts, and once more 7 of hearts next to 7 of clubs. Write X for the total score.*

Writing A_i, $i = 1, \ldots, 51$, for the event that there is a match of values for cards i and $i+1$, we have $\mathbb{P}(A_i) = 3/51$: given a card, there are 51 cards left, of which 3 have the same value.

a) Compute $\mathbb{E}X = 51(3/51) = 3$

b) Compute $\mathrm{Var}X$. This is a long variance-covariance expansion.

The trivial part is 51 variance terms, each with value $p - p^2$, $p = 3/51$.

The easy ingredient is

$$q = \mathbb{E}(A_1 A_2) = (3/51)(2/50)$$

contributing a strictly negative value for $q - p^2 = \mathrm{cov}(A_i, A_j)$ when $|i - j| = 1$, with 100 such terms.

The harder part is

$$r = \mathbb{E}(A_1 A_3)$$

to get a different nonzero value for $r - p^2 = \mathrm{cov}(A_i, A_j)$ when $|i - j| > 1$, with $51^2 - 51 - 100$ such terms.

A candidate value for r is $r = (3/51)(2/50)(1/49) + (3/51)(48/50)(3/49)$, corresponding to the first 4 cards having values of the form either aaaa, or else aabb with a different from b.

c) Compute $\mathbb{P}(X = 39) = (13!)(4!)^{13}/(52!)$. For this to happen, the cards should be stacked with all the same values next to each other:

$$aaab\ bbbb\ cccc\ \ldots mmmm$$

(13 groups of 4) Note that the the 4 values in each of the 13 groups can be permuted independently of one another, whence $(4!)^{13}$ term.

d) *In the line below, circle the number that you think is the closest to the value* $\mathbb{P}(X = 0)$ *and briefly explain your choice.*

$$\frac{1}{1000}, \frac{1}{500}, \frac{1}{100}, \frac{1}{50}, \mathbf{\frac{1}{20}}, \frac{1}{10}, \frac{1}{5}, \frac{1}{2}.$$

By Poisson approximation, $\mathbb{P}(X = 0) \approx e^{-3} \approx 1/20$.

3. *Let* $S_0, S_1, S_2, \ldots$ *be a simple symmetric random walk, i.e.* $\mathbb{P}(S_i - S_{i-1} = 1) = \mathbb{P}(S_i - S_{i-1} = -1) = 1/2$, *with independent increments. Let* $T = \min\{n > 0 : S_n = 0\}$ *be the hitting time to zero. Write* $\mathbb{P}_a$ *for probabilities for the walk starting with* $S_0 = a$.

a) What does the reflection principle say about $\mathbb{P}_a(S_n = i, T \le n)$, for $a > 0$, and $i, n \ge 0$?

Answer: $\mathbb{P}_a(S_n = i, T \le n) = \mathbb{P}_0(S_n = a + i)$.

b) What does the reflection principle say about $\mathbb{P}_a(S_n \ge i, T > n)$, for $a > 0$, and $i, n \ge 0$? [Hint: telescoping series]

Answer:

$$\mathbb{P}_a(S_n = j, T > n) = \mathbb{P}_0(S_n = j - a) - \mathbb{P}_0(S_n = j + a)$$

then sum up over j.

c) For fixed $a > 0$, give asymptotics for $\mathbb{P}_a(T > n)$ as $n \to \infty$. [HINT: Stirling's formula is that $n! \sim \sqrt{2\pi n}\,(n/e)^n$.]

Answer: since

$$\mathbb{P}_0(S_n = a + i) = 2^{-n}\binom{n}{\frac{n+a+i}{2}}$$

(the probability is zero if the fraction is not an integer), use part a) and Stirling to get

$$\mathbb{P}_a(T > n) \sim a\sqrt{2/(\pi n)}$$

d) Simplify, for fixed $a > 0$,

$$\lim_{n\to\infty} \frac{\mathbb{P}_{a+1}(T > n)}{\mathbb{P}_a(T > n)}.$$

Answer: from part c), get $(a + 1)/a$.