Ideas for solutions.

1. *(a) There are $n = 100$ balls, labeled 1 to n, and these are thrown into n boxes, also labeled 1 to n; all n^n outcomes are equally likely. Whenever ball i lands in box j, and $|i-j| \le 1$, a point is scored, so the total score, call it X, can take on any value from 0 to n. Note: ball 1 in box n scores nothing, and ball n in box 1 scores nothing. Compute exactly, and simplify, as either an expression in n, or a decimal: $E(X)$ and $Var(X)$.*

(b) Pick the closest approximation: in part (a), P($X = 0$) is close to 1, 1/3, 1/20, 1/100. You may reason informally, but you must describe your reasoning, which should involve the approximate distribution of X.

(c) Change the story in (a) to: there are $n = 100$ cards, numbered 1 to n, and they are placed in slots 1 to n around a circle at random, so that all $n!$ outcomes are equally likely. Whenever card i is placed in position i or $i + 1$, a point is scored, so the total score, call it Y, can take on any value from 0 to n. Here for card n, since the cards are in a circle, "position $n+1$" should be interpreted as position 1. Compute exactly, and simplify, as either an expression in n, or a decimal: $E(Y)$ and $Var(Y)$.

In part (a), each ball has probability $1/n$ to land in a given box, and balls land independently of one another; in particular, several balls can land in the same box. There are two ways to score in boxes 1 and n, and three ways to score in all other boxes. Thus, we have the sum of two Bernoulli $2/n$ and $(n-2)$ Bernoulli $3/n$, all independent. The mean is then $2(2/n)+(n-2)(3/n) = 3-(2/n)$. The variance is $2(2/n)(1-(2/n))+(n-2)(3/n)(1-(3/n)) = (3n^2 - 11n + 10)/n^2$. With mean approximately equal to three, the Poisson approximation gives the answer in (b) as $e^{-3} \approx 1/20$.

In part (c), each card has exactly two ways to score, and the same probability $1/n$ to land in any particular slot, so the mean score is exactly two. To compute the variance, write X_i for the indicator that ball i scores, with $p := EX_i = 2/n$

For the variance covariance expansion, $\sum cov(X_i, X_j)$ has

a) n terms where $i = j$, each term contributes $cov = p - p^2$.

b) $2n$ terms where i, j are adjacent. There are 3 ways for both to score, so each term contributes $cov = 3/(n(n-1)) - p^2$.

c) $n(n-3)$ terms where i, j are distinct and not adjacent. There are 4 ways for both to score, so each term contributes $cov = 4/(n(n-1)) - p^2$.

Adding up and simplifying leads to the final answer $2 - \frac{2}{n-1}$.

2. *(a) A coin comes up heads with probability $p \in [0, 1]$; it is tossed independently n times, and X is the total number of heads. Simplify the generating function*

$$G(s) = E(s^X),$$

and show how derivatives of G can be used to calculate $E(X)$ and $E(X^2)$.

(b) Random variables $U, U_1, U_2, \ldots, U_n$ are independent, and uniformly distributed in [0,1]. Let

$$Y = \sum_{1}^{n} 1_{\{U_i < U\}}$$

be the sum of indicators, counting how many of the U_i are less than U. Simplify the generating function

$$H(s) = E(s^Y),$$

and then simplify the ratio, $P(Y = 2)/P(Y = 1)$. HINT: conditionally on $U = p$, this Y is distributed as the X in part (a).

For (a), the answer is straightforward:

$$G(s) = G(s;p) = \sum_{k=0}^{n} \binom{n}{k} (sp)^k (1-p)^{n-k} = \big(1 + (s-1)p\big)^n.$$

The general definition of the (probability) generating function G_X of an integer-valued random variable X implies

$$E(X) = G_X'(1), \; E(X^2 - X) = G_X''(1).$$

After conditioning on $U = p$ and integrating, we get the answer to part (b):

$$H(s) = \int_0^1 G(s;p)dp = \frac{s^{n+1} - 1}{(n+1)(s-1)} = \frac{1 + s + \ldots + s^n}{n+1}.$$

The answer shows that Y is uniform on $\{0, 1, \ldots, n\}$, and, in particular,

$$\frac{P(Y=2)}{P(Y=1)} = 1.$$

3. *Let $X_1, \ldots, X_n$, $Y_1, \ldots, Y_m$ be iid uniform on $(0,1)$.*
(a) Compute the probability density function of $\max(Y_1, \ldots, Y_m)$.
(b) Compute the probability density function of

$$\frac{\max(X_1, \ldots, X_n)}{\max(Y_1, \ldots, Y_m)}.$$

The distribution function of the max of m iid uniforms is x^m, $x \in (0,1)$ (all m of them must be less than x), and the density is mx^{m-1}.

The answer for (b) is cu^{n-1}, $0 < u < 1$ and c/u^{m+1}, $u > 1$, where $c = mn/(m+n)$.

The solution for (b) is straightforward from the formula for the density of the ratio of two independent random variables; a complete solution should include a picture, such as the image of the unit square under the corresponding transformation.