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Algebra Exam Spring 2018

Problem 1. Prove that a group of order 72 cannot be simple.

Solution. Let G be a group of order 72. Note that $72 = 9 \cdot 8 = 3^2 \cdot 2^3$. Now, by the Sylow Theorems, $n_3 \equiv 1 \mod 3$ and $n_3 | 8$, so $n_3 = 1, 4$.

Assume G is simple. By the Sylow Theorems, Sylow 3-subgroups are conjugates and G can act on $\text{Syl}_3(G)$ the set of Sylow 3 subgroups by conjugation. Note that since $n_3 \neq 1$, $n_3 = 4$ and so $|\text{Syl}_3(G)| = 4$.

This induces a homomorphism $\varphi : G \to S_4$ where $\varphi(g) = \sigma_g$ which is the conjugation map

$$\begin{aligned}\sigma_g : \text{Syl}_3(G) &\to \text{Syl}_3(G)\\ P_3 &\mapsto gP_3g^{-1}\end{aligned}$$

Since G is simple, $\ker \varphi$ must be trivial, since kernels are normal subgroups.

However, then S_4 has an isomorphic copy of G inside it, which is not possible since $|S_4| = 4! = 24 < |G| = 72$.

This is a contradiction and so G cannot be simple.

***Note that $n_3 = 4$ could still be possible, however, in this case, the kernel of the homomorphism induced by the conjuation action cannot be trivial. ✌

Problem 2. Say that a group G is uniquely p-divisible if the p-th power map sending $x \in G$ to x^p is bijective. Show that if G is a finitely generated uniquely p-divisible *abelian* group, then G is finite and has order coprime to p.

Solution. Assume G is abelian and finitely generated. Then by the fundamental theorem of finitely generated abelian groups,

$$G \cong \mathbb{Z}^n \oplus (\mathbb{Z}_{p_1^{\alpha_1}})^{n_1} \oplus \cdots \oplus (\mathbb{Z}_{p_k^{\alpha_k}})^{n_k}$$

where p_i are primes, and the α_i are distinct.

Now, G is uniquely p divisible, and so if φ_p is the p^{th} power map, then φ_p is bijective. However, this is only possible if φ_p is bijective in each coordinate.

Let π_l be the projection homorphism to the l^{th} coordinate.

However, then we can restrict φ_p to the l^{th} coordinate to get that $\pi_l \circ \varphi_p|_{l^{\text{th}} \text{ coordinate}}$ is also bijective for all l.

Since φ_p is certainly not a surjective map restricted to $\mathbb{Z}$, $n = 0$. Namely, $|G| < \infty$.

Furthermore, restricting to an automorphism of $\mathbb{Z}_{p_i^{\alpha_i}}$, we immediately get that $p \neq p_i$. Else, there would exist an element x of order p in $\mathbb{Z}_{p_i^{\alpha_i}}$ and so $\varphi_p(x) = x^p = e$. making φ_p not injective on that coordinate.

Thus, $p \neq p_i$ for all i, and so G must have order coprime to p. ✌

Problem 3. Let $\mathbb{Q}$ be the field of rational numbers and consider $f(x) = x^8 + x^4 + 1 \in \mathbb{Q}[x]$. Write E for a splitting field for $f(x)$ over $\mathbb{Q}$ and set $G = \text{Gal}(E/\mathbb{Q})$. Find $|E : \mathbb{Q}|$ and determine the Galois group G up to isomorphism. If $\Omega \subset E$ is the set of roots of $f(x)$, find the number of orbits for the action of G on Ω.

Solution. Let $u = x^4$. Then $f(u) = u^2 + u + 1$, so

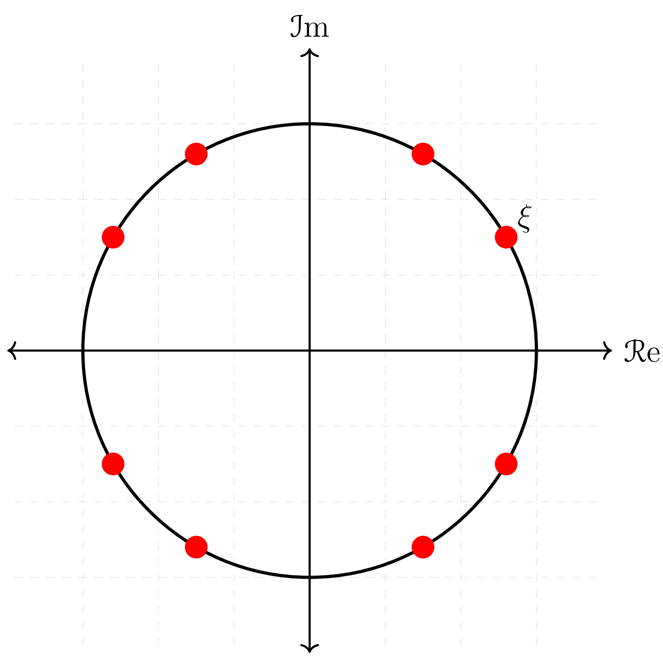
$$\begin{aligned} u^2 + u + 1 &= 0 \\ \implies u &= \frac{-1 \pm \sqrt{1-4}}{2} \\ &= \frac{-1 \pm \sqrt{3}i}{2} \\ &= e^{i2\pi/3}, e^{i4\pi/3} \end{aligned}$$

Now, if u is a root of $u^2 + u + 1$, then x is a 4^{th} root of u. At this point, we can note that the roots are all distinct and so E is the splitting field of a separable polynomial so it is a Galois extension of $\mathbb{Q}$ and so $G = \text{Gal}(E/\mathbb{Q})$ exists.

Now, $\xi = e^{i\pi/6}$ is a primitive root of $e^{i2\pi/3}$ since the four roots are

$$\xi = e^{i\pi/6}, \xi^4 = e^{i2\pi/3}, \xi^7 = e^{i7\pi/6}, \xi^{10} = e^{5i\pi/3}$$

and since $(e^{2i\pi/3})^2 = e^{i4\pi/3}$, we have that ξ actually generates all the roots of $f(x)$. Now, we note two things, first, if z is a root of $f(x)$ then $-z$ is also a root. Furthermore, if z is a root, then $\overline{z}$ is also a root. Thus, starting with u and $\overline{u}$, we can get that the roots are



Namely, the roots are

$$\xi, \xi^2, \xi^4, \xi^5, \xi^7, \xi^8, \xi^{10}, \xi^{11}.$$

Therefore, $E = \mathbb{Q}(\xi)$. Now, $\xi^6 = -1$ and so ξ satisfies $g(x) = x^6 + 1$

$$\begin{aligned} x^6 + 1 &= (x^2)^3 + 1 \\ &= (x^2 + 1)(x^4 - x^2 + 1) \end{aligned}$$

since ξ does not satisfy $x^2 + 1$, it must satisfy $x^4 - x^2 + 1$, which is irreducible over $\mathbb{Q}$ by the quadratic formula. Therefore,

$$[E : \mathbb{Q}] = 4.$$

Thus, by the Fundamenal Theorem of Galois theory, $|G| = 4$. Since there are only two groups of order 4 up to isomorphism, G is abelian and it is either isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$.

Finally, we check whether G has any elements of order 4.

Let $\sigma, \tau \in G$ be defined by $\sigma(\xi) = \overline{\xi}$ and $\tau(\xi) = -\xi$. Both of these are clearly well defined by the computation for the roots of f and both maps have order 2, so if they are not equal, then G must be $\mathbb{Z}_2 \times \mathbb{Z}_{\not\neq}$ since $\mathbb{Z}_4$ has only one element of order 4. However,

$$\overline{\xi} = e^{-i\pi/6} = e^{11i\pi/6} \qquad -\xi = e^{i(\pi/6+\pi)} = e^{i7\pi/6}$$

and these are clearly not equal, so

$$G = \{\text{Id}, \sigma, \tau, \sigma\tau\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Finally, using the image or via direct calculation, we see that the orbits are exactly $\{\xi^i, \sigma \cdot \xi^i, \tau \cdot \xi^i, \sigma\tau \cdot \xi^i\}$ where the action is defined by $g \cdot \xi^i = g(\xi^i)$ for $g \in G$.

Now, since τ fixes

has 2 orbits, namely

$$\{\xi, \xi^5, \xi^7, \xi^{11}\} \qquad \{\xi^2, \xi^4, \xi^8, \xi^{10}\}.$$

✌

Problem 4. Show that a 10-dimensional $\mathbb{C}$-algebra necessarily contains a non-zero nilpotent element (hint: what can you say about the Jacobson radical of such an algebra?).

Solution. There is something very wrong with this question. $\mathbb{C}^{10}$ is a ten-dimensional $\mathbb{C}$-algebra which clearly contains no non-zero nilpotent elements.

However, note that if A is a 10-dimensional $\mathbb{C}$-algebra, then A is artinian (finite dimensional) and so $J(A)$ is nilpotent.

Thus, if $J(A) \neq (0)$, then A will contain a non-zero nilpotent element, namely, an $x \in J(A)$. ✌

Problem 5. Consider the algebra $A := \mathbb{C}[M_n(\mathbb{C})]$ of polynomial functions of the ring of $n \times n$ matrices $M_n(\mathbb{C})$. Consider the polynomial functions defined by the formula $P_{ij}(X) := (X^n)_{ij}$. Let $I \subset A$ be the ideal defined by P_{ij}, $1 \leq i, j \leq n$. Describe the variety $V(I)$ and use your description to show that $I \neq \sqrt{I}$.

Solution. We note that if $P_{ij}(B) = 0$ for all i, j, for some B, then B is a nilpotent matrix of order less than or equal to n.

Therefore, $V(I)$ is exactly the set of tuples which form a nilpotent matrix of degree $\leq n$.

Now, by nullstellenzatz part II, there is a one-to-one correspondence between $V(I)$ and $\sqrt{I}$.

Let X be a nilpotent matrix of degree $2n > n$. Then X^2 is nilpotent of degree n and so X^2 is satisfied by all points in the variety $V(I)$. Therefore, by Nullstellenzatz, so there exists a k so $(X^2)^k = X^{2k} \in I$. Since there is a positive integer $l = 2k$ for which $X^l \in I$, $X \in \sqrt{I}$ by definition.

Now, if $X \in I$, then X itself must be satisfied by every point in $V(I)$, however, $X^n \neq 0$ since X has order $> n$, so this contradicts that $X \in I$.

Thus, $I \neq \sqrt{I}$. ✌

Problem 6. Is the ring $k[x,y]/(y^2-x^3)$ integrally closed in its field of fractions?

Solution. Let $R = k[x,y]/(y^2-x^3)$. First, we note that $\frac{y}{x}$ is certainly in the field of fractions of $k[x,y]$ and so it is in the field of fractions of R. Furthermore,

$$\left(\frac{y}{x}\right)^2 = \frac{y^2}{x^2} = \frac{x^3}{x^2} = x$$

and so

$$\left(\frac{y}{x}\right)^2 - x = 0 \qquad \text{in } R.$$

Clearly, $g(z) = z^2 - x$ is a monic polynomial in $R[z]$ and so if we can show that $\frac{y}{x} \notin R$, then we have that R is NOT integrally closed.

Now, let

$$\begin{aligned} \varphi : k[x,y] &\to k[t] \\ x &\mapsto t^2 \\ y &\mapsto t^3 \end{aligned}$$

then $(y^2-x^3) \subset \ker(\varphi)$. Now, suppose $f(x,y) \in \ker\varphi$. Then we can apply division algorithm to f and y^2-x^3 and write

$$f(x,y) = (y^2-x^3)f_1(x,y) + r_1(x,y).$$

where r_1 has only y terms of degree 1 and x terms of degree 2 or less.

Now,

$$f(t^2,t^3) = r_1(t^2,t^3) = 0$$

so $r_1 \in \ker\varphi$ as well. However, then we can write

$$r_1(x,y) = a_1 + a_2x + a_3x^2 + a_4y + a_5xy + a_6x^2y.$$

However, then

$$\begin{aligned} r_1(t^2,t^3) &= 0 \\ &= a_1 + a_2t^2 + a_3t^4 + a_4t^2 + a_5t^5 + a_6t^7. \end{aligned}$$

and so $a_i = 0$ for all i.

Namely, $r_1 = 0$ and so $f(x,y) \in (y^2-x^3)$.

Finally, $R \cong k[t^3,t^2]$ is not integrally closed since t is a root of $h(z) = z^2 - t^2 \in k[t^3,t^2][z]$ but $t \notin k[t^3,t^2]$.

Therefore, R is not integrally closed. ✌

Problem 7. Suppose R is a commutative (unital) ring, M and N are R-modules, and $f : M \to N$ is an R-module homomorphism. Show that f is surjective if and only if, for every prime ideal $\mathfrak{p} \subset R$, the induced map $f_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ of modules over $R_\mathfrak{p}$ is surjective.

Solution. We will actually show something stronger, that localization is exact.

Let P be a prime ideal of R. Then $M_P = S^{-1}M$ is the localization of of M over the localization ring $S^{-1}R$ where $S = R\backslash P$.

Namely,

$$f_P : M_P \to N_P$$
$$\frac{m}{s}m \mapsto \frac{f(m)}{s}$$

$\boxed{\Longrightarrow}$ Assume f is surjective. Then, for all $n \in N$, there exists an $m \in M$, with $f(m) = n$.

Then if $\frac{n}{s} \in N_P$, there exists $m \in M$ so $f(m) = n$ so

$$f_P\left(\frac{m}{s}\right) = \frac{f(m)}{s} = \frac{n}{s}$$

and so f_P will be surjective for all prime ideals P.

***Similarly, if f is injective, and

$$f_P\left(\frac{m}{s}\right) = \frac{f(m)}{s} = 0$$

then there is a $t \in S$ so $tf(m) = f(tm) = 0$ so $tm \in \ker f = (0)$ so m is torsion and $tm = 0 \in M$.
However, this is exactly what it means for $\frac{m}{s} = 0$ in M_P.

$\boxed{\Longleftarrow}$ Assume f_P is surjective for all prime ideals P.

Now, we prove a claim.

Claim 1. An R-module T is trivial if and only if T_P is trivial for all prime ideals P.

Proof. $\boxed{\Longrightarrow}$ This is clear.

$\boxed{\Longleftarrow}$ Assume $T_P = S^{-1}T$ is trivial for all prime ideals P.

Let $x \in T$. Assume $x \neq 0$. Then let $I = \{r \in R \,|\, rx = 0\}$. Then $I \neq R$ since $1x = x \neq 0$. Furthermore, I is an ideal.

Thus, there exists a maximal (prime) ideal P so $I \subset P$.

Now, $T_P = (0)$, so since $\frac{x}{1} \in T_P$, there exists an $s \in S$ so $sx = 0 \in T$. However, $s \in S = R \backslash P$ so namely, $s \notin I$.

This is a contradiction and so no such x can exist. Namely, T is trivial. ✌

Now, let $g : N \to N/f(M)$ be the quotient map. Then since

$$(N/f(M))_P = N_P/f(M)_P = N_P/f_P(M_P)$$

$g_P : N_P \to N_P/f(M)_P$ is well defined.

Now, we clearly have a right exact sequence

$$M_P \xrightarrow{f_P} N_P \xrightarrow{g_P} N_P/f_P(M_P) \longrightarrow 0$$

since if $\frac{n}{s} \in \ker g_P$ then

$$g_P\left(\frac{n}{s}\right) = \frac{g(n)}{s} \in f_P(M_P) = \text{Im}(f).$$

And clearly the reverse is also true.

However, f_P is surjective for all P, and so g_P is the zero map for all P so $N_P/f_P(M_P)$ is trivial for all P. Therefore, by **Claim 1**, $N/f(M)$ is trivial and so f is surjective.

***Similarly, assume f_P is injective for all P. Assume there is some $0 \neq m \in M$ with $f(m) = 0$. Then

$$f_P\left(\frac{m}{1}\right) = \frac{f(m)}{1} = 0$$

so f_P sends $\frac{m}{1}$ to $0 \in M_P$ for all P.
Let $I = \{r \in R \,|\, rm = 0\}$ which is the torison ideal of the element m. Then $I \neq R$ since $1m = m \neq 0$. Also, I is contained in some maximal (prime) ideal P.
Now, $f_P(m) = 0$ and f_P is injective, so there is some $t \in S$ with $tm = 0 \in M$. However, if $t \in S$, then $t \notin P$ and so $t \not{i}nI$. This is a contradiction. and so there exists $t \in S$
Thus, no such nonzero $m \in M$ exists and f is injective.

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