Kayla Orlinsky

Algebra Exam Spring 2016

Problem 1. Let R be a Noetherian commutative ring with 1 and $I \neq 0$ an ideal of R. Show that there exist finitely many nonzero prime ideals P_i of R (not necessarily distinct) so that $\prod_i P_i \subset I$ (Hint: consider the set of ideals which are not of that form).

Solution. Let

$$S = \{J \mid J \text{ does not contain a finite product of nonzero prime ideals}\}$$

the set of ideals of R not of the form described.

If S is empty, then we are done, so assume not.

Then S is partially ordered by inclusion. Furthermore, any ordered chain of elements of S contains a maximal element in S, namely the union of all ideals in the chain. Since a union (including infinite union) of ideals is an ideal, and since none of the ideals in the chain contain a finite product of primes, their union won't either.

Therefore, by Zorn's Lemma, S contains a maximal element J.

Now, let $xy \in J$. If $x \notin J$, then $J + xR$ is an ideal strictly larger than J.

If $J + xR = R$ then

$$yR = yJ + yxR = J + xyR \subset J$$

since $xy \in J$ so $xyr \in J$ for all $r \in R$.

However, then $y \in J$ and this implies J is prime, clearly a contradiction.

Assume $J + xR \neq R$. Similarly, $J + yR \neq R$. Now, because $J \subset J + xR$ and $J \subset J + yR$, $J + xR$ and $J + yR$ must both contain a finite product of nonzero prime ideals. If not, then this contradicts the maximality of J.

Therefore,

$$(J + xR)(J + yR) = J + xyR \subset J$$

and so J again contains a finite product of nonzero prime ideals.

This is again a contradiction, and so J cannot exit. Namely, S must be empty. ✌

Problem 2. Describe all groups of order 130: show that every such group is isomorphic to a direct sum of dihedral and cyclic groups of suitable orders.

Solution. Let G be a group of order 130. Note that $130 = 2 \cdot 5 \cdot 13$. This gives one abelian group

$$\mathbb{Z}_{130}.$$

By the Sylow theorem, $n_{13} = 1$ the number of Sylow 13 subgroups. This is because $n_{13} | 2 \cdot 5$ and $n_{13} \equiv 1 \mod 13$ by the Sylow Theorems and so $n_{13} \neq 2, 5, 10$. Thus, $n_{13} = 1$.

So G has a normal Sylow 13-subgroup, P_{13}.

Therefore, $P_5 P_{13}$ is a subgroup of G and since it has index 2, it is normal.

However, by **Fall 2011: Problem 5 Claim 3**, P_5 is normal in $P_5 P_{13}$ so P_5 is normal in G.

$\boxed{\varphi : P_2 P_5 \to \mathrm{Aut}(P_{13})}$ $\varphi : P_2 P_5 \to \mathrm{Aut}(P_{13}) \cong \mathbb{Z}_{12}$. If $P_2 \cong \langle a \rangle$, $P_5 \cong \langle b \rangle$, and $P_{13} \cong \langle c \rangle$, then the only possible non-trivial homomorphism sends $(a, 0) \mapsto 6$ since this is the only element of $\mathbb{Z}_{12}$ of order 2, the inversion map. Namely, we get multiplication relation, $aca^{-1} = \varphi(a)(c) = c^{-1}$.

This gives a possible group

$$G \cong \langle a, b, c \,|\, a^2 = b^5 = c^{13} = 1, ab = ba, bc = cb, ac = c^{-1}a \rangle.$$

$\boxed{\varphi : P_2 P_{13} \to \mathrm{Aut}(P_5)}$ $\varphi : P_2 P13 \to \mathbb{Z}_4$. This gives one possible homomorphisms, again, inversion $\varphi(a, 0) = 2$.

This gives multiplication $aba^{-1} = \varphi(a)(b) = b^{-1}$ so we get

$$\langle a, b, c \,|\, a^2 = b^5 = c^{13}, ac = ca, bc = cb, ab = b^{-1}a \rangle.$$

$\boxed{\varphi : P_2 \to \mathrm{Aut}(P_5 P_{13})}$ $\varphi : P_2 \to \mathbb{Z}_4 \times \mathbb{Z}_{12}$. Then there are now three possible homomorphisms, $\varphi(a) = (2, 0), (0, 6), (2, 6)$. Clearly the first two we will have already seen before since they define the relations $aba^{-1} = b^{-1}$, $aca^{-1} = c$, and $aba^{-1} = b$, $aca^{-1} = c^{-1}$ respectively.

Thus, the only new relation gives

$$G \cong \langle a, b, c, \,|\, a^2 = b^5 = c^{13} = 1, bc = cb, ab = b^{-1}a, ac = c^{-1}a \rangle.$$

$\boxed{\varphi : P_5 \to \mathrm{Aut}(P_2 P_{13})}$ If $P_2 P_{13}$ is normal in G then we can examine $\varphi : P_5 \to \mathbb{Z}_1 \times \mathbb{Z}_{12} \cong \mathbb{Z}_{12}$. Clearly, all such homomorphisms are trivial.

$\boxed{\varphi : P_{13} \to \mathrm{Aut}(P_2 P_5)}$ If $P_2 P_5$ is normal in G then we can examine $\varphi : P_{13} \to \mathbb{Z}_1 \times \mathbb{Z}_4 \cong \mathbb{Z}_4$. Clearly, all such homomorphisms are trivial.

This concludes all possible groups.

Finally, we note that

$$\langle a,b,c\,|\,a^2=b^5=c^{13}=1, ab=ba, bc=cb, ac=c^{-1}a\rangle \cong \langle a,c\|\,a^2=c^{13}=1, ac=c^{-1}a\rangle\times\mathbb{Z}_5 \cong D_{26}\times\mathbb{Z}_5.$$

Where D_{26} is the dihedral group of 26 elements. Similarly, we obtain

$$\langle a,b,c\,|\,a^2=b^5=c^{13}, ac=ca, bc=cb, ab=b^{-1}a\rangle \cong D_{10}\times\mathbb{Z}_{13}.$$

Finally, if $bc=cb, ab=b^{-1}a, ac=c^{-1}a$ then bc is an element of order 65 since $bc=cb$ and

$$abc=b^{-1}ac=b^{-1}c^{-1}a=c^{-1}b^{-1}a=(cb)^{-1}a.$$

Therefore, this exactly describes

$$\langle a,b,c,\,|\,a^2=b^5=c^{13}=1, bc=cb, ab=b^{-1}a, ac=c^{-1}a\rangle \cong D_{130}.$$

Finally, we have

$$\mathbb{Z}_2\times\mathbb{Z}_5\times\mathbb{Z}_{13}$$

$$D_{26}\times\mathbb{Z}_5$$

$$D_{10}\times\mathbb{Z}_{13}$$

$$D_{130}$$

✌

Problem 3. Let $f(x) = x^{12} + 2x^6 - 2x^3 + 2 \in \mathbb{Q}[x]$. Show that $f(x)$ is irreducible. Let K be the splitting field of $f(x)$ over $\mathbb{Q}$. Determine whether $\mathrm{Gal}(K/\mathbb{Q})$ is solvable.

Solution. This problem is very similar to **Fall 2015: Problem 7**.

$f(x)$ is irreducible over $\mathbb{Q}$ by Eisenstein's criterion with $p = 2$. Then p does not divide the leading coefficient, p divides all other coefficients, and p^2 does not divide the constant term.

Since irreducible implies separable in fields of characteristic 0, we have that K is the splitting field of a separable polynomial so it is a Galois extension.

Let a, b, c, d be the roots of $u^4 + 2u^2 - 2u + 2$. Then letting $u = x^3$ we see that the roots of $f(x)$ are the third roots of a, b, c, d.

Namely, if L is the splitting field of $u^4 + 2u^2 - 2u + 2$, then K/L is clearly a radical extension of L, so it suffices to check if L is a radical extension of $\mathbb{Q}$.

Now, since $4u^3 + 4u - 2$ is negative for all $u < \alpha$ where $\alpha \in (0, 1)$ and positive for all $u > \alpha$, we have that $u^4 + 2u^2 - 2u + 2$ has a single minimum for some value between 0 and 1.

Since $u^4 + 2u^2 + 2 > 2 > 2u$ for any value in $(0, 1)$, we have that $u^4 + 2u^2 - 2u + 2 > 0$ and so this polynomial has no real roots.

Therefore, it has two sets of complex conjugate roots, $a, \overline{a}$ and $b, \overline{b}$.

Since $u^4 + 2u^2 - 2u + 2$ is irreducible by Eisenstein with $p = 2$, we have that $L = \mathbb{Q}(a, \overline{a}, b, \overline{b})$ is also Galois over $\mathbb{Q}$. Thus, $H = \mathrm{Gal}(K/L)$ is normal in $G = \mathrm{Gal}(K/\mathbb{Q})$ and $\mathrm{Gal}(L/\mathbb{Q}) = G/H$.

Now, each third rood in K clearly has minimal polynomial $x^3 - a$, $x^3 - b$, $x^3 - \overline{a}$, $x^3 - \overline{b}$ over L. These are irreducible since factoring would force a linear term to appear over L, and L does not contain any third roots of $a, b, \overline{a}, \overline{b}$.

So $[K : L] \leq 3^{12}$. Specifically, since each of these is irreducible over L, $[K : L] = 3^r$ for some $r \leq 12$.

However, then clearly H has order 3^r and so it must be solvable. This is because p-groups have non-trivial centers, and so recursively, we could obtain a chain by examining $H/Z(H)$, $H/Z(H)/Z(H/Z(H))$, etc.

Finally, $a, b, \overline{a}, \overline{b}$ all have minimal polynomial of degree 4 over $\mathbb{Q}$, so $[G : H] \leq 4^4$, so G/H is solvable.

Therefore, since H is normal in G, and H is solvable and G/H is solvable, then G is solvable. ✌

Problem 4. Determine up to isomorphism the algebra structure of $\mathbb{C}[G]$ where $G = S_3$ is the symmetric group of degree 3. (Recall that $\mathbb{C}[G]$ is the group algebra of G which has basis G and the multiplication comes from the multiplication on G).

Solution. By Artin Wedderburn, $\mathbb{C}[S_3]$ is semi-simple of dimension 6 so

$$\mathbb{C}[S_3] \cong \mathbb{C}^a \oplus (M_2(D))^b$$

where D is a division ring over $\mathbb{C}$.

Note that $M_n(D)$ cannot appear for $n > 2$ since the dimension of the algebra is 6 and $M_3(D)$ has dimension $3^2 = 9$. For the same reason, there can be only one copy of $M_2(D)$. Namely, $b = 0, 1$.

Furthermore, by Frobenius, the only division ring over $\mathbb{C}$ is $\mathbb{H}$, and since $\mathbb{C} \subset Z(\mathbb{C}[S_3])$ is contained in the center of the algebra (definition of algebra), we have that $\mathbb{H}$ cannot appear in the decomposition. Also, $D = \mathbb{C}$ since any central division ring over an algebraically closed field is the base field.

Finally, since S_3 is non commutative, $b = 1$ and so

$$\mathbb{C}[S_3] \cong \mathbb{C}^2 \oplus M_2(D).$$

Note that this follows, since S_3 has 3 conjugacy classes and so it has 3 simple components.

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Problem 5. If F is a field and $n > 1$ show that for any nonconstant $g \in F[x_1, ..., x_n]$ the ideal $gF[x_1, ..., x_n]$ is not a maximal ideal of $F[x_1, ..., x_n]$.

Solution. Let $R = F[x_1, ..., x_n]$ and $I = (g) = gR$. Then if R/I is a field, we have that $f + I$ has an inverse in R/I for all $f \in R$.

Namely, there exists $h + I$ such that $(f + I)(h + I) = fh + I = 1 + I$. Thus, there exists $r \in R$ so

$$fh + gr = 1 \in R.$$

Thus, for all $f \in R$, there exists $h, r \in R$ so $fh + gr = 1$ in R.

However, then $I + fR = R$ for any $f \in R$.

Let K be the algebraic closure of F and $J = I + fR$ be an ideal of R. Then by Nullsetellensatz, $1 \in J$ if and only if $V(J)$ is empty as a subset of K^n.

Since we have already seen that $I + fR = R$ for any $f \in R$, we have that $1 \in J$ for any $f \in R$.

However, then $V(J) = \varnothing$ in K^n for any $f \in R$. That is, g and f share no zeros, where f is any polynomial.

This forces g to be a nonzero constant. ✌

Problem 6. Let F be a field and let P be a submodule of $F[x]^n$. Suppose that the quotient module $M : F[x]^n/P$ is Artinian. Show that M is finite dimensional over F.

Solution. Note that if M is finite dimensional as a module over F, then M is an F-vector space.

Now, let $(0, ..., 0, x, 0, ..., 0) + P$ be an element of M, where x is in the i^{th} position. Then we have a decreasing chain,

$$(0, ..., 0, x, 0, ..., 0) + P \supset (0, ..., 0, x^2, 0, ..., 0) + P \supset (0, ..., 0, x^3, 0, ..., 0) + P \supset \cdots$$

that, since M is artinian, must terminate after a finite number of steps.

Namely, $(0, ..., 0, x^{m_i}, 0, ..., 0) \in P$ for some m_i.

Since this holds for every position of the tuple, we get that

$$\bigcup_{i=1}^{n} \{(0, ..., 0, x, 0, ..., 0), (0, ..., 0, x^2, 0, ..., 0), ..., (0, ..., 0, x^{m_i-1}, 0, ..., 0)\}$$

forms an F-basis for M. Since this set is clearly finite, we have that M has a finite basis over F and so M is a finite dimensional F-vector space. ✌