ALGEBRA QUALIFYING EXAM FALL 2010

Do all six problems. Each problem is worth 4 points and partial credit may be awarded.

1. Use Sylow's Theorems to show that any group of order $(99^2 - 4)^3$ is solvable.

2. For any finite group G and positive integer m, let $n_G(m)$ be the number of elements g of G that satisfy $g^m = e_G$. If A and B are finite abelian groups so that $n_A(m) = n_B(m)$ for all m, show that as groups $A \cong B$.

3. If $g(x) = x^5 - 2 \in Q[x]$, for Q the field of rational numbers, compute the Galois group of a splitting field L over Q of $g(x)$. How many subfields of L containing Q are Galois over Q?

4. Let P be a minimal prime ideal in the commutative ring R with 1: that is, if Q is a prime ideal in R and if $Q \subseteq P$, then $Q = P$. Show that each $x \in P$ is a zero divisor in R.

5. Set $R = C[x_1, \ldots, x_n]$ with $n \geq 3$ and C the field of complex numbers. For any subset $S \subseteq R$, let $\mathcal{V}(S) = \{\alpha \in C^n \mid g(\alpha) = 0$ for all $g \in S\}$. Consider the ideal I of R defined by $I = (x_1 \cdots x_{n-1} - x_n, x_1 \cdots x_{n-2}x_n - x_{n-1}, \ldots, x_2 \cdots x_n - x_1)$, so the generators of I are obtained by subtracting each x_i from the product of the others. Show that there are fixed positive integers s and t so that for each $0 \leq i \leq n$, $(x_i^s - x_i)^t \in I$. (Hint: Consider the product of the generators of I.)

6. Let R be a right artinian algebra over an algebraically closed field F. Show that R is algebraic over F of bounded degree. That is, show there is a fixed positive integer m so that for any $r \in R$ there is a nonzero $g_r(x) \in F[x]$ with $g_r(r) = 0$ and with $\deg g \leq m$.

$$\begin{array}{c} R \\ | \\ F \end{array}$$

① $(99^2-4)^3 = (99-2)^3(99+2)^3 = 97^3 \cdot 101^3$ (97 + 101 prime)

$\rightsquigarrow$ Solvable by Burnside.

Sylow: $r_{97} \equiv 1 \pmod{97}$ & $r_{97} \mid \frac{97^3 \cdot 101^3}{97^3} \Rightarrow r_{97} = 1$

and same for $r_{101} \Rightarrow$ All Sylow subgps are normal

$\Rightarrow$ grp is dir. prod. of Syl. subgps

Both Sylow subgps cyclic prime ords, hence abelian, hence

$G = P \oplus Q$ abelian since P, Q are. $\therefore$ Solvable

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④ $P \subseteq R$ min prime ?

$s \in P$, R_P localization, PR_P only prime ideal in R_P $\to$

so $\frac{s}{1}$ nilpotent in R_P ?

Hence $\exists\, t \in R \setminus P$ s.t. $ts^n = 0$, hence s zero divisor

?

: ~~Algebra~~ - Fall 2016 :

① Use Sylow to prove that $(99^2-4)^3$ is solvable.

See that $(99^2-4)^3 = (99-2)(99+2)^3 = 97^3 101^3$, 97 & 101 prime

Sylow : $r_{97} \equiv 1 \pmod{97}$ & $r_{97} \mid 101^3 \Rightarrow r_{97} = 1, \cancel{101}, \cancel{101^2}, \cancel{101^3}$

$\Rightarrow r_{101} \equiv 1 \pmod{101}$ & $r_{101} \mid 97^3 \Rightarrow r_{101} = 1, \cancel{97}, \cancel{97^2}, \cancel{97^3}$

$\Rightarrow$ Both Sylow 97-sg & 101-sg are normal.

$\Rightarrow G = S_{97} \oplus S_{101}$

But $S_{97} \trianglelefteq G$ is solvable since all p-grps are solvable ; on the other hand, $G/S_{97} \cong S_{101}$ is also solvable (p-grp) $\Rightarrow$ G solvable.

② For $|G|<\infty$, let $n_G(m) = \#\{g \in G : g^m = e\}$

A, B finite abelian groups so that $n_A(m) = n_B(m)$ for all m, show that $A \cong B$.

Let $A \cong \mathbb{Z}_{p^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{e_k}} \oplus$ (other primes)

we have $n_B(p) = n_A(p) = kp = \#$ of p-primary summands.

Hence $B \cong \mathbb{Z}_{p^{f_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{f_k}} \oplus$ (other primes), i.e. for each prime p, A & B have the same # of p-primary summands.

It is only left to show that the exponents are the same, i.e. $\{e_i\} = \{f_i\}$. Let $l_m = |\{e_i \geq m\}|$; then see that:

$$n_A(p^2) = n_A(p) + l_2 p^2,$$

$$n_A(p^3) = n_A(p^2) + l_3 p^3, \; \ldots, \; n_A(p^m) = n_B(p^{m-1}) + l_m p^m.$$

Hence $n_B(p^m) = n_A(p^m) \Rightarrow n_B(p^{m-1}) + l_m^{(B)} p^m = n_A(p^{m-1}) + l_m^{(A)} p^m$

$$\Rightarrow l_m^{(B)} p^m = l_m^{(A)} p^m$$

$$\Rightarrow |\{f_i \geq m\}| = |\{e_i \geq m\}| \text{ for all } m$$

$$\Rightarrow \{f_i\} = \{e_i\}.$$

Hence for all p, A & B have the same # p-primary summands & same exponents, hence $A \cong B$.

(3) $g(x) = x^5 - 2 \in \mathbb{Q}[x]$ & let $L/\mathbb{Q}$ be splitting field.
Compute $Gal(L/\mathbb{Q})$ & find # subfields of L that are Galois over $\mathbb{Q}$.

$g(x)$ is irreducible by Eisenstein with $p=2$ + g separable since over $\mathbb{Q}$
$\Rightarrow L/\mathbb{Q}$ Galois extension, use FToGT.
Let $\omega^5 = 1$ and then the roots of g are $-\omega^j \sqrt[5]{2}$, $j = 1, \dots, 5$;
hence $L = \mathbb{Q}(\sqrt[5]{2}, \omega)$ splitting field; now consider the diagram:

$$\begin{tikzcd}
 & L = \mathbb{Q}(\sqrt[5]{2}, \omega) & \\
\mathbb{Q}(\sqrt[5]{2}) \arrow[ur, dash, "4"] & & \mathbb{Q}(\omega) \arrow[ul, dash, "5"'] \\
 & \mathbb{Q} \arrow[ul, dash, "5"] \arrow[ur, dash, "4"'] &
\end{tikzcd}$$

$\Rightarrow |Gal(L/\mathbb{Q})| = 20$

$\leftarrow$ cyclotomic ext, so $Gal(\mathbb{Q}(\omega)/\mathbb{Q}) \cong \mathbb{Z}_4 = \langle \tau \rangle$

min poly for $\sqrt[5]{2}$ is $x^5 - 2$, so $Gal(\mathbb{Q}(\sqrt[5]{2})/\mathbb{Q}) \cong \mathbb{Z}_5 = \langle \sigma \rangle$

note: $\mathbb{Q}(\omega)/\mathbb{Q}$ is Galois, hence there is index 4 norm. subgroup of $Gal(L/\mathbb{Q})$

Now see that: $\bullet \begin{cases} \sigma: \omega \mapsto \omega \\ -\sqrt[5]{2} \mapsto -\omega^j \sqrt[5]{2} \end{cases}$ $\bullet \begin{cases} \tau: -\sqrt[5]{2} \mapsto -\sqrt[5]{2} \\ \omega \mapsto \omega^k \end{cases}$ are the gen. of the Galois group.

Consider the conjugation:
$$\tau\sigma\tau^{-1}(\sqrt[5]{2}) = \tau\sigma(\sqrt[5]{2}) = \tau(\omega^j \sqrt[5]{2}) = \omega^{jk}\sqrt[5]{2}$$
$$\tau\sigma\tau^{-1}(\omega) = \tau\sigma(\omega^{1/k}) = \tau(\omega^{1/k}) = \omega$$

Let $j=1$ (any elt in $\mathbb{Z}_5$ gen.)
$k=3$ (3 in $\mathbb{Z}_4$ gen.) $\Longrightarrow \tau\sigma\tau^{-1} = \sigma^3$

Hence, $\underline{Gal(L/\mathbb{Q}) = \langle \sigma, \tau : \sigma^5 = \tau^4 = 1, \tau\sigma\tau^{-1} = \sigma^3 \rangle}$

Subfields:

$$\begin{tikzcd}
 & G = Gal(L/\mathbb{Q}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\boxed{\langle \sigma \rangle = \mathbb{Z}_5} & \mathbb{Z}_4 = \langle \tau \rangle \arrow[d, dash] & \boxed{\langle \tau^2, \sigma \rangle} \\
 & \mathbb{Z}_2 = \langle \tau^2 \rangle &
\end{tikzcd}$$

order 10, hence index 2, hence normal.

$\sigma\tau^2\sigma^{-1} = \sigma\tau\tau\sigma^4 = \dots = \tau\sigma\tau\sigma \neq \tau^2$
$\Rightarrow \langle \tau^2 \rangle$ not normal

not Galois since not normal, hence $\mathbb{Z}_4$ not normal s.g.

(also $\langle \tau\sigma \rangle$ another Sylow 2-s.g. hence $\langle \tau \rangle$ not unique)

normal s.g. since $\mathbb{Q}(\omega)/\mathbb{Q}$ Galois

$\boxed{\text{fields } L \supseteq K \supseteq \mathbb{Q} \text{ Galois over } \mathbb{Q} \text{ are } L^{\mathbb{Z}_5} \text{ \& } L^{\langle \tau^2, \sigma \rangle}}$

④ P minimal prime ideal in R commutative
(i.e., if Q prime in R & $Q \subseteq P$, then $Q = P$)
Show: every $x \in P$ is a zero divisor:

Let $P \subseteq R$ be minimal prime and consider the localization R_P.
The prime ideals in R_P are of the form QR_P where $Q \subseteq P$ is prime,
hence since P is minimal, PR_P is the unique prime ideal in
R_P. Now recall that, since R commutative,

$$Nil(R_P) = \sqrt{(0)} = \bigcap_{\substack{P \supseteq (0) \\ P \text{ prime} \\ \text{in } R_P}} P = \bigcap_{\substack{P \text{ prime} \\ \text{in } R_P}} P = PR_P .$$

Hence the ideal PR_P consists of nilpotent elements;
choose $x \in P$ and then $\frac{x}{1} \in PR_P$ has $\left(\frac{x}{1}\right)^n = 0$ for some n, minimal.

$\Rightarrow 0 = x^n t \in R$ for some $t \notin P$, $t \neq 0$

$\Rightarrow x(x^{n-1}t) = 0$ in R, but $x^{n-1}t \neq 0$ since n minimal,

hence x is a zero divisor.

(5) $R=\mathbb{C}[x_1,\dots,x_n]$, $n\geq 3$, Consider $I=(x_1\cdots x_{n-1}-x_n,\ x_1\cdots x_{n-2}x_n-x_{n-2},\ \dots,\ x_2\cdots x_n-x_1)$

Show $\exists\, s,t$ such that $(x_i^s-x_i)^t\in I$

See that $(x_i^s-x_i)^t\in I \iff (x_i^s-x_i)\in\sqrt{I}=Id(Var(I))$

$\iff (x_i^s-x_i)$ vanishes on $Var(I)$.

Suppose $(y_1,\dots,y_n)\in Var\, I$ $(y_i\neq 0\ \forall i)$. Then:

$$\begin{aligned} y_1&=y_2\cdots y_n\\ y_2&=y_1y_3\cdots y_n\\ &\ \vdots\\ y_n&=y_1y_2\cdots y_{n-1}\end{aligned} \Rightarrow y_1\cdots y_n=(y_1\cdots y_n)^{n-1}$$

$$\Rightarrow 1=(y_1\cdots y_n)^{n-2}$$

$$\Rightarrow 1=y_i^{n-2}(y_1\cdots\hat{y_i}\cdots y_n)^{n-2}=y_i^{n-2}(y_i)^{n-2}$$

$$=y^{2i-4}\Rightarrow y_i=y_i^{2i-3}$$

$$\Rightarrow y_i^{2n-3}-y_i=0\ \forall y_i$$

Hence, letting $s=2n-3$, $x_i^s-x_i$ vanishes on $Var(I)$, hence $x_i^s-x_i\in\sqrt{I}$.

Now, since $\sqrt{I}=\langle f_1,\dots,f_m\rangle$ finitely generated and let $k=\max\{m : f_j^m\in I,\ m \text{ minimal}\}$, for any $f=\sum_{i=1}^m g_if_i$, we get $f^{mk}=\left(\sum_{i=1}^m g_if_i\right)^{mk}\in I$, hence let $t=mk$ and therefore $(x_i^s-x_i)^t\in I$.

⑥ R artinian algebra over alg. closed field F and R/F algebraic extension. Show: R algebraic over F of bdd degree (i.e. $\exists m$ s.t. $\forall r \in R$, $\exists g_r(x) \in F[x]$ with $g_r(r)=0$ & $\deg g_r \leq m$)

Semisimple case: Suppose R is semisimple. Then by Art-Wedd:
$R \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$. Each $M_{n_i}(D_i)$ contains a copy of D_i hence D_i are division algebras over F, hence $F \subseteq D_i$.

Choose $\alpha \in D_i$ & consider $F(\alpha)/F$. R/F is algebraic, hence $[F(\alpha):F] < \infty$; but $F(\alpha) = F$ since algebraically closed $\Rightarrow \alpha \in F$ $\Rightarrow D_i \subseteq F \Rightarrow D_i = F$; hence $R \cong \bigoplus_{i=1}^{m} M_{n_i}(F)$

Since $R \cong M_{n_1}(F) \oplus \cdots \oplus M_{n_k}(F)$, $a \in R$ has the form $a = a_1 + \cdots + a_k$, $a_i \in M_{n_i}(F)$. Since a_i is a matrix over a field, it will have minimal polynomial p_i with $\deg p_i \leq n_i$ (as a consequence of Cayley-Hamilton). So then:

$$p(x) = \prod_{i=1}^{k} p_i(x - \sum_{j \neq i} a_j) \text{ will have } p(a) = 0.$$

$$\text{and } \deg p = \deg \prod_{i=1}^{k} p_i \leq \sum_{i=1}^{k} \deg p_i \leq \sum_{i=1}^{k} n_i$$

So then let $\underline{m = \sum n_i}$.

General case: Consider $R/J(R)$. It is Jacobson s.s. by construction and artinian since R is $\Rightarrow R/J(R)$ is semisimple.

Now we can apply the preceding case, i.e. $\exists M$ such that for all $a \in R$, $\exists f \in F[x]$, $\deg f < M$, where $f(a) \in J(R)$.
But R Artinian $\Rightarrow J(R)$ nilpotent $\Rightarrow \exists n$ s.t. $J(R)^n = 0$, hence $f(a)^n = 0$; see that $\deg f^n \leq Mn$, hence let $\underline{m = Mn}$