(c) $A \trianglelefteq G$ & $\frac{|G|}{|A|} = p$

$\{\text{Sylow } p\text{-sg } S \;\; A \leq S \leq G\} = \frac{N_G(P)}{N_G(Q)}$

$A \trianglelefteq G$.

(1) (a) $P \leq A$

$g \in G$, $gPg^{-1} \leq A$ since A normal.

hence gPg^{-1} also Sylow subgroup of A

thus $\exists x \in A$ s.t. $xPx^{-1} = gPg^{-1}$

since P, gPg^{-1} conj.

Algebra Qualifying Examination, Spring 2008

Directions

This exam consists of 7 problems. Please do 6 of them and show your work. If you are using a well-known result in your proof, please refer to it by name. Good Luck!

1. Let G be a finite group with A a normal subgroup of G. Let P be a Sylow p-subgroup of A. (a) If $g \in G$, show that $gPg^{-1} = xPx^{-1}$ for some $x \in A$. (b) Prove that $G = AN_G(P)$. (c) Prove that if $[G : A] = p$, then the number of Sylow p-subgroups of G that contain P is equal to $[N_G(P) : N_G(Q)]$ where Q is a Sylow p-subgroup of G containing P.

2. Let G be a finite group of order n. Let $f : G \to S_n$ be the regular representation of G. (a) Show that the image of f is contained in the alternating group if and only if the Sylow 2-subgroup of G is not cyclic. (b) Use (a) to show that if G has a cyclic nontrivial Sylow 2-subgroup, then G contains a normal subgroup N of odd order (hint: show first that G has a normal subgroup of index 2).

3. Let A be a (commutative) integral domain that is not a field and let K be the quotient field of A.

(a) Prove that K is not finitely generated as an A-module. Hint: use Nakayama's lemma.

(b) Can K ever be finitely generated as an A-algebra?

4. Let K be a field and p be a prime number that is not equal to the characteristic of K. Assume that K does not contain a primitive p-th root of unity. Let $a \in K^*$ that is not a p-th power and let b be an

(b) $N_G(P) = \{g : gPg^{-1} = P\}$

$g \in G$, then $\exists x \in A$ st. $xPx^{-1} = gPg^{-1}$

~~$\Rightarrow P = x^{-1}gPg^{-1}x$~~

$\Rightarrow xpx^{-1} = gsg^{-1}$

i.e. $\exists\, xpx^{-1} = gsg^{-1}$

$\Rightarrow g = xpx^{-1}gs^{-1}$

$x \in A$, $p \in P$ $\Rightarrow xp \in A$ since $P \leq A$

$xgs^{-1}Psg^{-1}x$

$xgPg^{-1}x = P$ by assump

$\Rightarrow xgs^{-1} \in N_G(P)$

element of an separable closure of K with $b^p = a$. Consider the field $L = K(\zeta, b)$, where ζ is a primitive p-th root of unity. Prove that L/K is a Galois extension and that $Gal(L/K)$ is isomorphic to the group of 2 by 2 matrices

$$\begin{pmatrix} 1 & r \\ 0 & s \end{pmatrix}$$

where $r \in \mathbb{Z}/p\mathbb{Z}$ and $s \in (\mathbb{Z}/p\mathbb{Z})^*$. Is this group abelian?

5. Let $R = \mathbb{C}[x, y]$ and consider the two ideals $I = (2x + y)$ and $J = (x^2 - y)$. (a) show that I and J are both prime ideals of R, and that each of them is the intersection of all of the maximal ideals containing it.

(b) Consider the ideal $I + J$. Is it a prime ideal?

(c) Same question for $I \cap J$.

(hint: you can get a lot of intuition for this problem by thinking about the analogous varieties in $\mathbb{R}^2$).

6. (i) Let A be a finitely generated abelian group. If ℓ is a prime number, let $A[\ell]$ denote the set of elements of A that are killed by ℓ. Then $A[\ell]$ and $A/\ell A$ are finite groups, say of orders n_1 and n_2, respectively. Express the difference $n_2 - n_1$ in terms of other invariants of A.

(ii) If A is an abelian group such that the groups $A[\ell]$ and $A/\ell A$ are finite, is A necessarily finitely generated? Give a proof or a counterexample.

7. Let R be a (left) Artinian ring which is an algebra over the field k. Assume that every $r \in R$ is algebraic over k, with a minimal polynomial of the form

$$x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0,$$

such that either a_0 or a_1 is non-zero. Show that R is a direct sum of division rings.

$\zeta^p = 1$

$\tau^{\frac{p-1}{2}} \cdot \tau^{\frac{p-1}{2}} = \tau^{p-1} = 1$

$\frac{p(p-2)+1}{4}$

$\left(\frac{p-1}{2}\right)^2 = \frac{p^2-2p+1}{4}$

$(2x+y)(x^2-y)$

$= 2x^3 - 2xy + yx^2 - y^2$

$I \cap J \supseteq IJ$

$ij \in IJ$

$\cancel{ij \in IJ}$

$\in J, \; i \in I \setminus J$

$j \in J \setminus I$

$ij \in IJ.$

$\cancel{ij \in I}$

Algebra - Spring 2008 :

① $|G|<\infty$, $A \trianglelefteq G$, $P \leq A$ p-subgroup of A.

(a) If $g \in G$, show that $gPg^{-1} = xPx^{-1}$ for some $x \in A$.

$P \leq A$, and A is normal, so $gPg^{-1} \subseteq A$; but $|gPg^{-1}| = |P|$, hence gPg^{-1} is also a Sylow p-s.g. of A.

Therefore, since by Sylow all Sylow p-s.g.'s are conjugate, P and gPg^{-1} are conjugate in A, hence $\exists x \in A$ s.t. $xPx^{-1} = gPg^{-1}$.

(b) $G = AN_G(P)$:

$N_G(P) = \{t \in P : tPt^{-1} = P\}$

Choose $g \in G$, $s \in P$. Then, by pt.(a), $\exists x \in A$, $r \in P$ s.t. $gsg^{-1} = xrx^{-1}$.

$$\Rightarrow g = xrx^{-1}gs^{-1} = xr(x^{-1}gs^{-1})$$

$\left.\begin{matrix} x \in A, \\ r \in P \leq A \end{matrix}\right\} \Rightarrow xr \in A$

$x^{-1}gs^{-1}P(x^{-1}gs^{-1})^{-1} = x^{-1}gs^{-1}Psg^{-1}x$
$s \in P \rightarrow = x^{-1}gPg^{-1}x = P$

Recall $xPx^{-1} = gPg^{-1} \Rightarrow P = x^{-1}gPg^{-1}x$

Hence, $x^{-1}gs^{-1} \in N_G(P)$.

Therefore $G = AN_G(P)$.

(c) If $[G:A] = p$, then # of Sylow p-s.g.'s of G that contain A is equal to $[N_G(P) : N_G(Q)]$ where Q is a Sylow p-s.g. of G containing P.

② $|G|=n$, $f: G\to S_n$ reg. rep'n. [Cf. #2 FALL 09]

(a) $f(G)\subseteq A_n \iff$ Sylow 2-s.g. of G is not cyclic

Prove by contrapositive, i.e. $f(G)\nsubseteq A_n \iff$ Sylow 2-s.g. is cyclic.

($\Leftarrow$) Suppose Sylow 2-s.g. is cyclic, i.e. $|G|=2^m k$, k odd & G has elt order 2^m. Under the rep'n: $f: G\to S_n$
$g\mapsto \gamma_g(h)=gh$; γ_g has fixed pt $\iff g=e$,
hence γ_g consists of $\frac{|G|}{\text{ord}(g)}$ distinct cycles of length ord(g).
Now let $x\in G$ be order 2^m; then γ_x is k cycles of length 2^m, i.e. an odd # of even length cycles. Even length cycles have odd # of transpositions, and odd·odd = odd, hence $\gamma_x\notin A_n$.
Therefore $f(G)\nsubseteq A_n$.

($\Rightarrow$) Suppose now that $f(G)\nsubseteq A_n$; i.e. there exists an element $y\in G$ s.t. $f(y)=\gamma_y$ can be factored into $\frac{2^m k}{\text{ord}(y)}$ cycles of length ord(y) where $\frac{2^m k}{\text{ord}(y)}$ odd & ord(y) even (so an odd # of transpositions, hence $\gamma_y\notin A_n$), hence $2^m \mid \text{ord}(y)$. Now $\langle y\rangle$ is a cyclic s.g. of order ord(y), but by Cauchy there must be a subgroup of order 2^m since $2^m\mid\text{ord}(y)$; but this is a subgroup of a cyclic grp, hence also cyclic, hence $\langle y\rangle$ has a cyclic s.g. order 2^m, hence G has cyclic s.g. order 2^m, hence the Sylow 2-s.g. of G is cyclic.

(b) use (a) to show if G has cyclic Sylow 2-s.g., then $\exists N\trianglelefteq G$ of odd order

G has cyclic Sylow 2-s.g., so by (a) $f(G)\nsubseteq A_n$, hence $GA_n\cong S_n$ and therefore: $2=|S_n/A_n|=|GA_n/A_n|\cong|G/G\cap A_n|$ by 2nd iso thm, hence G has an index 2 subgroup, call it $H\le G$. Then $|H|=2^{m-1}k$. Since $\exists x\in G$ w/ ord$(x)=2^m$, we also have that ord$(x^2)=2^{m-1}$. Since γ_{x^2} has $\frac{2^m k}{2^{m-1}}$ (even) cycles length 2^{m-1}, we have that $\gamma_{x^2}\in A_n$, hence $x^2\in H$, hence H has cyclic Sylow 2-s.g., hence H has index 2 subgroup by same argument. We may inductively produce a chain of index 2 subgroups until getting to $N\le H\cong G\cap A_n$ of odd order. If y is odd order then γ_y has even # cycles, hence in A_n. Therefore N is all odd order elts in G, hence normal.

(3) A comm integral domain that is not a field. Let $K = \mathrm{Frac}(A)$.

(a) Show that K is not fin-gen as an A-module

Suppose K is a fin-gen A-module. A not field $\Rightarrow \exists M$ max ideal in A.

Consider the localization at M:

$$A_M = \{\tfrac{a}{b} : a \in A,\ b \in A \setminus M\} / \tfrac{a}{b} \sim \tfrac{c}{d} \iff s(ad - bc) = 0 \text{ for some } s \in A \setminus M.$$

Then $J(A_M) = MA_M$, the unique maximal ideal (since A_M is local).

Now extend scalars of K from A to A_M: $K_M = K \otimes_A A_M$ (K_M an A_M-module)

We'll now apply Nakayama's Lemma; see that $(MA_M)K_M = K_M$:

$$\underset{\substack{\cap \\ K_M}}{\tfrac{c}{d} \otimes \tfrac{a}{b}} = \underset{\substack{\cap \\ MA_M}}{\left(\tfrac{ka}{b}\right)} \underset{\substack{\cap \\ K_M}}{\left(\tfrac{c}{kd} \otimes 1\right)} \Rightarrow K_M \subseteq (MA_M)K_M \Rightarrow K_M = (MA_M)K_M$$

So by Nakayama, $K_M = 0$,

Now, if $K_M = 0$ for all maximal ideals $M \subseteq A$, then $K = 0$.

(b) Can K be fin-gen as an A-algebra?

yes; consider $A = \mathbb{Z}_{(2)} = \{\tfrac{a}{b} \in \mathbb{Q} \mid a, b \in \mathbb{Z},\ b > 0,\ \gcd(2, b) = 1\}$

Then $\mathrm{Frac}(A) = \mathbb{Q}$, and furthermore $\mathrm{Frac}(A) = A[\tfrac{1}{2}]$, hence fin-gen alg

④ K field, p prime, $p \neq \mathrm{char}(K)$; K doesn't contain p^{th} rt of unity.
Let $a \in K^*$ not a p^{th} power & let b be an elt of sep. closure of K such that $b^p = a$.
Consider the field $L = K(\zeta, b)$ where $\zeta^p = 1$ primitive.
Prove: L/K ① Galois ext & $\mathrm{Gal}(L/K) \cong \left\{ \begin{pmatrix} 1 & r \\ 0 & s \end{pmatrix} : r \in \mathbb{Z}_p,\ s \in \mathbb{Z}_p^* \right\}$ ②
(is it abelian?)

$$\begin{tikzcd}
 & K(\zeta, b) \arrow[dl, dash, "p"'] \arrow[dr, dash, "p-1"] & \\
K(\zeta) & & K(b) \\
 & K \arrow[ul, dash, "p-1"] \arrow[ur, dash, "p"'] &
\end{tikzcd}$$

∘ Consider the polynomial $x^p - a \in K[x]$.
Clearly $L = K(\zeta, b)$ is a splitting field for it
f is separable $\Rightarrow$ L/K ① is Galois

$\hookrightarrow$ cyclotomic, so $\mathrm{Gal}(K(\zeta)/K) = \mathbb{Z}_{p-1}$

$\gg$ $x^p - a$ is the minimal polynomial for b over $K(\zeta)$, hence
$[K(\zeta, b) : K(\zeta)] = p$ $\Rightarrow$ $|\mathrm{Gal}(L/K)| = p(p-1)$.

cyclotomic

$K(b)/K$ is a Galois extension, hence $\mathrm{Gal}(K(\zeta,b)/K(b)) \cong \mathbb{Z}_{p-1}$
is a normal subgroup of $\mathrm{Gal}(L/K)$: $\mathbb{Z}_{p-1} \trianglelefteq \mathrm{Gal}(L/K)$.

Consider: $\begin{cases} \sigma : b \mapsto \zeta b \\ \quad\ \zeta \mapsto \zeta \end{cases}$ $\quad \tau : b \mapsto b$, $\zeta \mapsto \zeta^n$ $\quad$ which generate $\mathrm{Gal}(L/K)$.

$\sigma^p = 1$ $\qquad \tau^{p-1} = 1$

$\tau\sigma\tau^{-1}(b) = \tau\sigma(b) = \tau(\zeta b) = \zeta^n b = \sigma^n(b)$ $\Rightarrow \tau\sigma\tau^{-1} = \sigma^n$
$\tau\sigma\tau^{-1}(\zeta) = \tau\sigma(\zeta^{1/n}) = \tau(\zeta^{1/n}) = \zeta = \sigma^n(\zeta)$

- Now see the maps: $\sigma \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\sigma^k \mapsto \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k \in \mathbb{Z}_p$

$\tau \mapsto \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$, $\tau^m \mapsto \begin{pmatrix} 1 & 0 \\ 0 & n^m \end{pmatrix}$, $n \in \mathbb{Z}_p^* \cong \mathbb{Z}_{p-1}$

So then $\overset{\tau}{\begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}} \overset{\sigma}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} \overset{\tau^{-1}}{\begin{pmatrix} 1 & 0 \\ 0 & n^{-1} \end{pmatrix}} = \begin{pmatrix} 1 & 1 \\ 0 & n \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & n^{-1} \end{pmatrix} = \begin{pmatrix} 1 & n^{-1} \\ 0 & 1 \end{pmatrix} = \underset{\sigma^n}{\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}}$

$\Rightarrow n = n^{-1} \Rightarrow n = \frac{p-1}{2}$.

⑤ $R = \mathbb{C}[x,y]$ & consider two ideals $I = (2x+y)$ & $J = (x^2-y)$.

(a) Show I & J are prime ideals & each is the intersection of all max ideals containing it.

- I & J are prime since $2x+y$ & x^2-y are irreducible. ✓
- See that $\sqrt{I} = Id(Var\, I) = Id\left(\bigcup_{\alpha \in I} \{a_\alpha\}\right)$

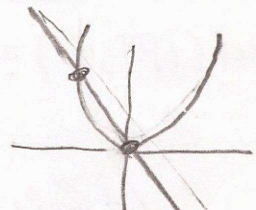
$$= \sqrt{\bigcap_{\alpha\in I} Id(a_\alpha)} = \sqrt{\bigcap_{\alpha\in I} (X - a_\alpha)} = \bigcap_{\alpha\in I}\sqrt{(X-a_\alpha)} = \bigcap_{\alpha\in I}(X-a_\alpha)$$

max ideal in R by weak Nullst.

So $I = \sqrt{I}$ is intersection of max ideals.

(b) Is $I+J$ prime?

consider the graph of the varieties:



Clearly $Var\, I \cap Var\, J$ is 2 points, say $\{\alpha, \beta\}$.

But now $Var\, I+J = Var\, I \cap Var\, J = \{\alpha, \beta\}$, and recall that

$\mathbb{C}[x,y]/\sqrt{I} = \mathbb{C}^{|Var\, I|}$

If $I+J$ were prime then $I+J = \sqrt{I+J}$, hence

$$\mathbb{C}[x,y]/_{I+J} \cong \mathbb{C}[x,y]/_{\sqrt{I+J}} \cong \mathbb{C}^2$$

But $\mathbb{C}^2$ is not an integral domain, hence $I+J$ not prime.

(c) Is $I \cap J$ prime?

$I \cap J = \langle (x^2-y)(2x+y) \rangle$, hence $(x^2-y)(2x+y) \in I \cap J$.

But $J = (x^2-y)$, hence $(2x+y) \notin J$, hence $(2x+y) \notin I \cap J$.

Likewise, $I = (2x+y)$, hence $(x^2-y) \notin I$, hence $(x^2-y) \notin I \cap J$.

Hence $I \cap J$ not prime.

⑥ (i) A fin-gen abelian group, ℓ prime.

Let $A[\ell] = \{m \in A : \ell m = 0\}$.

Then $A[\ell]$ & $A/\ell A$ are finite grps order n_1 & n_2 resp.

Express the number $n_2 - n_1$.

$$A \cong \mathbb{Z}^k \oplus \mathbb{Z}_{\ell^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{\ell^{e_m}} \oplus \overset{P}{\overset{\shortparallel}{(\text{other primes})}}$$

$n_1 = |A[\ell]| = \ell m$

and see that $A/\ell A \cong (\mathbb{Z}/\ell\mathbb{Z})^k \oplus \mathbb{Z}_{\ell^{e_1}}/\ell\mathbb{Z}_{\ell^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{\ell^{e_m}}/\ell\mathbb{Z}_{\ell^{e_m}} \oplus P/\ell P$.

Note that since $\gcd(|P|, \ell) = 1$, $\ell P = P$, hence $P/\ell P \cong 0$,

and also see that $\mathbb{Z}_{\ell^{e_1}}/\ell\mathbb{Z}_{\ell^{e_1}} \cong \dfrac{\mathbb{Z}/(\ell^{e_1})}{\ell\mathbb{Z}/(\ell^{e_1})} \cong \mathbb{Z}/\ell\mathbb{Z}$,

hence $A/\ell A \cong (\mathbb{Z}/\ell\mathbb{Z})^{k+m}$, hence $n_2 = |A/\ell A| = \ell(k+m)$

and therefore $n_2 - n_1 = \ell(k+m) - \ell m = \ell k = \underline{\ell(\underbrace{\mathrm{rk} A}_{\text{free rank}})}$

(ii) If A abelian s.t. $A[\ell]$ & $A/\ell A$ are finite, is A necessarily finitely-generated?

Let $A \cong \mathbb{Z}^k \oplus \mathbb{Z}_{\ell^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{\ell^{e_m}} \oplus \left(\bigoplus_{i=1}^{\infty} \mathbb{Z}_{p^{f_i}}\right)$ where $\gcd(\ell, p) = 1$.

A is not finitely generated, but clearly $|A[\ell]| < \infty$ & $|A/\ell A| < \infty$