ALGEBRA QUALIFYING EXAM, Fall 2008

1. Let p, q be odd primes with $p > 7$ and $q > 8p$. Let G be a group of order $8pq$.
(a) Show that G has a normal subgroup of order pq.
(b) Show that G has a normal subgroup of index 2.
(c) Show that G has a nontrivial center.

2. Let $G = L_1 \times \ldots \times L_t$, for $t > 1$, where all of the L_i are simple groups.
(a) Assuming that all of the L_i are nonabelian, prove that the only normal subgroups of G are direct products of some subset of the L_i. (Hint: Let N be a normal subgroup of G and show that if the ith projection of N into L_i is nontrivial, then N contains L_i).
(b) Now suppose that all $L_i \cong L$, with L simple (possibly abelian). Show that there is no nontrivial proper subgroup of G which is invariant under all automorphisms of G. (Hint: Consider the abelian and nonabelian cases separately.)
(c) Suppose that $G = L \times L$ with L a nonabelian simple group. Let $D = \{(x, x) | x \in L\}$ be the diagonal subgroup. Show that D is a maximal subgroup of G.

3. Consider $f(x) = x^4 + x^2 + 9 \in \mathbb{Q}[x]$.
(a) Show that $f(x)$ is irreducible over $\mathbb{Q}$. (Hint: first show that the only possible factors are quadratic, and then see what happens when x is replaced by $-x$.)
(b) Find the Galois group of $f(x)$ over $\mathbb{Q}$.
(c) Describe the splitting field of f over $\mathbb{Q}$ and the intermediate fields.

4. Let R be a commutative Noetherian ring. Show that any surjective ring endomorphism $\phi: R \to R$ is an automorphism.
(Hint: consider the iterations $\phi, \phi^2, \phi^3, \ldots$.)

5. Let I be the ideal

$$I = (x^{37}y^{31}z^{29}t^{23}, x^3 + y^5, y^7 + z^{11}, z^{13} + t^{17}) \subset \mathbb{C}[x, y, z, t].$$

If $f(x, y, z, t)$ is any polynomial without constant term show that some power of f is in I.

6. Let A be a finite-dimensional algebra over $\mathbb{C}$. Show that if $x, y \in A$ such that $xy = 1$, then also $yx = 1$.

7. Let A, B, C be finitely generated modules over a PID R. Show that B is isomorphic to C if and only if $A \oplus B$ is isomorphic to $A \oplus C$.

Algebra – Fall '08:

① p, q odd primes, $p > 7$, $q > 8p$, $|G| = 8pq$.

(a) G has normal s.g. order pq

Sylow: $r_2 \equiv 1 \pmod 2$ & $r_2 \mid pq \Rightarrow r_2 = 1, p, q, pq$

$r_p \equiv 1 \pmod p$ & $r_p \mid 2^3 q \to r_p = 1, \cancel{2}, \cancel{4}, \cancel{8}, q, 2q, 2^2 q, 2^3 q$.

$r_q \equiv 1 \pmod q$ & $r_q \mid 2^3 p \Rightarrow r_q = 1, \cancel{2}, \cancel{4}, \cancel{8}, \cancel{2p}, \cancel{2^2 p}, \cancel{2^3 p}, \cancel{p}$

↳ Sylow q-s.g. is normal, $(q > 8p)$
let N be it.

Let S be a Sylow p-s.g.
Then NS is a s.g. of order pq; now recall rep'n on cosets:
$NS \leq G$, $[G : NS] = 2^3 \Rightarrow \exists\, \varphi : G \to S_8$ w/ $\ker\varphi \subseteq NS$.

Now, $|G/\ker\varphi| \mid |S_8| = 8!$ & $|G/\ker\varphi| \mid |G| = 8pq$.

$\Rightarrow |G/\ker\varphi| \mid \gcd(8!, 8pq) = 8$

Hence $|\ker\varphi| = 2^k pq$ for some k, but $\ker\varphi \subseteq NS$,
hence $|\ker\varphi| \mid pq$, hence $|\ker\varphi| = pq \Rightarrow \ker\varphi = NS$
$\Rightarrow$ NS normal.

(b) G has normal s.g. index 2

$|G| = 8pq = 2^3 pq$.

Recall NS is normal s.g.; by Cauchey we have s.g. Q of order $2^2 = 4$, Hence NSQ is a s.g. of order $2^2 pq$, hence $[G : NSQ] = 2$ & NSQ normal since index 2.

② $G = L_1 \times \cdots \times L_t$, $t > 1$, where all L_i are simple.

(a) Assuming all L_i are nonabelian, show only normal s.g.'s of G are products of the L_i.

Let $N \trianglelefteq G$ and consider φ_i, the i^{th} proj. onto L_i, i.e.

$$\varphi_i : G = L_1 \times \cdots \times L_i \times \cdots \times L_t \to L_i$$
$$(g_1, \dots, g_i, \dots, g_t) \mapsto g_i$$

Suppose that $\varphi_i(N) \neq e$, i.e. $\varphi_i(N) \leq L_i$. Since φ_i is surjective by defn, $\varphi_i(N) \trianglelefteq L_i \Rightarrow \varphi_i(N) = L_i$ since L_i simple.
$\Rightarrow L_i \subseteq N$.

Since the L_i are simple, for each i $\varphi_i(N) = L_i$ or $\{e\}$, hence $L_i \subseteq N$ precisely when $\varphi_i(N) \neq e$.

Let $L_{i_1}, \dots, L_{i_k}$ be the L_i s.t. $\varphi_{i_j}(N) = L_{i_j}$.

Then $L_{i_1} \times \cdots \times L_{i_k} \subseteq N$. Since the projection is trivial for all other L_i, no elt of the other L_i can be in N.
Therefore all elts of G are exhausted & $L_{i_1} \times \cdots \times L_{i_k} = N$, hence a normal s.g. must be a product of L_i's.

(b) Suppose $L_i = L$ simple $\forall i$. Show $\nexists$ nontrivial proper s.g. of G invariant under all automorphisms.

$G = L^t$; let $H \leq G$ and suppose invariant under all of $Aut(G)$. Therefore, inv. under all of $Inn(G)$, hence $H \trianglelefteq G$, hence $H = H_1 \times \cdots \times H_t$ where $H_i = L$ or 1; suppose $H_i = 1$ & $H_j = L$. Now H cannot be invariant under the reordering autom. switching the i & j coordinates. ↳(an outer autom)

(c) Suppose $G = L \times L$, L nonabel, simple.
Let $D = \{(x,x) : x \in L\}$; show D is maximal in G.

Suppose not; i.e., $\exists H$ s.t. $D \subsetneq H \subsetneq G$. By part (a), G has normal sbgs $L \times 1$ & $1 \times L$.

Now, $\underbrace{H \cap (L \times 1)}_{\shortparallel\, N} \trianglelefteq H$ is normal in H. and 1×2

and $\underline{N \cap D = 1}$ since $(L \times 1) \cap D = 1$

Now see that, for $(x,y) \in H$ we have:

$$H \ni (x,y) = \underbrace{(xy^{-1}, 1)}_{\in H \cap (L\times 1) = N} \underbrace{(y,y)}_{\in D \subseteq H}$$

since $(y^{-1}, y^{-1}) \in D \subseteq H$ and $(x,y) \in H$,
$(x,y)(y^{-1},y^{-1}) = (xy^{-1}, 1) \in H$; clearly in $L \times 1$

Hence, since $N \cap D = 1$, we now have $\underline{H = ND}$.

Now, choose $(b,1) \in N = H \cap (L \times 1) \Rightarrow (b,1) \in H$
$(g,1) \in L \times 1$

$$\text{so } \underbrace{\underset{D \subseteq H}{(g,g)}\ \underset{H}{(b,1)}\ \underset{D \subseteq H}{(g,g)^{-1}}}_{\in H} = \underset{H \cap (L \times 1) = N}{(gbg^{-1}, 1)} = \underset{L\times 1}{(g,1)}\ \underset{N}{(b,1)}\ \underset{L \times 1}{(g,1)^{-1}}$$

$\Downarrow$

N is normal in $L \times 1$;

But, $L \times 1$ is simple, hence $N = 1 \times 1$ or $L \times 1$

i.e. $(H \cap L \times 1) = 1 \times 1 \Rightarrow H = 1 \times K$ (some k)
or $(H \cap L \times 1) = L \times 1 \Rightarrow H = L \times 1$
} contradictions since $D \subseteq H$

③ $f(x) = x^4 + x^2 + 9 \in \mathbb{Q}[x]$.

(a) Show f is irred over $\mathbb{Q}$

See that $x^2 = \frac{-1 \pm \sqrt{1-4(1)(9)}}{2} = \frac{-1 \pm \sqrt{-35}}{2}$, hence no real roots, hence only quadratic factors over $\mathbb{Q}$.

Now, see that $f(-x) = x^4 + x^2 + 9 = f(x)$; and now:

$$f(x) = (x^2+ax+b)(x^2+cx+d)$$
$$f(-x) = (x^2-ax+b)(x^2-cx+d)$$

$\Rightarrow (x^2+ax+b)(x^2+cx+d) = (x^2-ax+b)(x^2-cx+d)$.

These must have the same irreducible factors over $\mathbb{Q}$, hence:

- Case 1: $(x^2+ax+b) = (x^2-ax+b) \Rightarrow a=0$

  $(x^2+cx+d) = (x^2-cx+d) \Rightarrow c=0$

  so $f(x) = (x^2+b)(x^2+d) = x^4 + (b+d)x^2 + bd$

  $\Rightarrow b+d = 1$, $bd = 9$, no sol'n, hence no factorization.

- Case 2: $(x^2+ax+b) = (x^2-cx+d)$, $(x^2+cx+d) = (x^2-ax+b)$ $\Rightarrow \begin{cases} (a+c)x = d-b = -(b-d) \\ (a+c)x = b-d \end{cases}$

  $\Rightarrow a+c = 0$ & $b=d$

  $f(x) = (x^2+ax+b)(x^2+cx+d)$

  $= (x^2+ax+b)(x^2-ax+b) = x^4 + (2b-a^2)x^2 + b^2$

  $\Rightarrow b^2 = 9$, $2b - a^2 = 1$

  So $b = 3$, $a^2 = 5$ no rational sol'n

  hence no factorization

Let $\alpha = \frac{-1+i\sqrt{35}}{2}$

$\bar{\alpha} = \frac{-1-i\sqrt{35}}{2}$

(b) Galois grp of f over $\mathbb{Q}$:

$E = \mathbb{Q}(\sqrt{\alpha}, \sqrt{\bar\alpha})$ is the splitting field; note that $\sqrt{\alpha\bar\alpha} = \sqrt{9} = 3$,

hence $\sqrt{\bar\alpha} = \frac{3}{\sqrt{\alpha}}$, hence $\sqrt{\bar\alpha} \in \mathbb{Q}(\sqrt{\alpha})$, so $E = \mathbb{Q}(\sqrt{\alpha})$

Hence now consider the extensions $\mathbb{Q} \subseteq \mathbb{Q}(\alpha) \subseteq \mathbb{Q}(\sqrt{\alpha})$.

$$\begin{array}{c} \mathbb{Q}(\sqrt{\alpha}) \\ | \\ \mathbb{Q}(\alpha) \end{array} \Big\} 2 \text{ since } x^2 - \alpha \text{ min poly of } \sqrt{\alpha} \text{ over } \mathbb{Q}(\alpha)$$

$$\begin{array}{c} \mathbb{Q}(\alpha) \\ | \\ \mathbb{Q} \end{array} \Big\} 2 \quad \text{since } \mathbb{Q}(\alpha) = \mathbb{Q}(i\sqrt{35}) \; (= \mathbb{Q}(\bar\alpha))$$

so $[\mathbb{Q}(\sqrt{\alpha}) : \mathbb{Q}] = 4$

Now consider automorphisms $\sigma: \sqrt{\alpha} \mapsto \sqrt{\bar\alpha}$; order 2

$\tau: \sqrt{\alpha} \mapsto -\sqrt{\alpha}$; order 2.

Two distinct order 2 elts in $|Gal(E/\mathbb{Q})| = 4$, hence

$Gal(E/\mathbb{Q}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

(c) Describe splitting field and intermediate fields

$\mathbb{Q}(\sqrt{\alpha})$ splitting field

$\langle \tau \rangle \trianglelefteq Gal(E/\mathbb{Q})$ fixes $\mathbb{Q}(\alpha)$

$\langle \sigma \rangle \trianglelefteq Gal(E/\mathbb{Q})$ fixes $\mathbb{Q}(\sqrt{\alpha} + \sqrt{\bar\alpha})$ ← degree 2 extension since $(\sqrt{\alpha} + \sqrt{\bar\alpha})^2$

$= \alpha + \bar\alpha + 2\sqrt{\alpha\bar\alpha}$

$= -1 + 2 \cdot 3 = 5 \in \mathbb{Q}$,

hence min poly is $\underline{x^2 - 5}$.

↑ only non-trivial subgroups, so:

$$\begin{tikzcd} & \mathbb{Q}(\sqrt{\alpha}) \arrow[dl, dash] \arrow[dr, dash] & \\ \mathbb{Q}(\alpha) \arrow[dr, dash] & & \mathbb{Q}(\sqrt{\alpha} + \sqrt{\bar\alpha}) \arrow[dl, dash] \\ & \mathbb{Q}. & \end{tikzcd}$$

(4) R comm, noeth.

show any surj. ring endo. $\varphi: R \to R$ is an autom.

WTS $\ker \varphi = 0$. Consider the chain of ideals

$$\ker \varphi \subset \ker \varphi^2 \subseteq \ker \varphi^3 \subseteq \cdots \subseteq \ker \varphi^n = \ker \varphi^{n+1}$$

which terminates at some n since R noetherian.

Consider $x \in \ker \varphi$; since $\varphi^n(R) = R$ (φ surjective), $\exists\, y \in R$ s.t. $x = \varphi^n(y)$, but then $0 = \varphi(x) = \varphi^{n+1}(y)$,

hence $y \in \ker \varphi^{n+1} = \ker \varphi^n \Rightarrow \varphi^n(y) = 0$

$\Rightarrow x = \varphi^n(y) = 0$

$\Rightarrow x = 0 \Rightarrow \ker \varphi = 0$.

(5) $I = (x^{37} y^{31} z^{29} t^{23},\ x^3 + y^5,\ y^7 + z^{11},\ z^{13} + t^{17}) \subseteq \mathbb{C}[x, y, z, t]$

If $f \in \mathbb{C}(x, y, z, t)$ is any poly w/o constant term, show some pwr of f is in I. (i.e. $f \in \sqrt{I}$)

Consider $Var(I)$. Since $x^{37} y^{31} z^{29} t^{25} \in I$, one coordinate of all members of $Var(I)$ must be zero.

Consider $\alpha = (x, y, z, t) \in Var(I)$:

$x = 0$ & $x^3 + y^5 \in I \Rightarrow y = 0$,

$y = 0$ & $x^3 + y^5,\ y^7 + z^{11} \in I \Rightarrow x = 0$ & $z = 0$

$z = 0$ & $y^7 + z^{11},\ z^{13} + t^{17} \in I \Rightarrow y = 0,\ t = 0$

$t = 0$ & $z^{13} + t^{17} \Rightarrow z = 0$

f

hence if one is zero, all of them are, hence $Var(I) = 0$

and therefore $Id(Var(I))$ = polynomials w/ no constant term,

but by nullstellensatz, $\sqrt{I} = Id(Var(I)) \Rightarrow f \in \sqrt{I}$

(6) A fin-dim'l alg. over $\mathbb{C}$.
Show that if $x, y \in A$ s.t. $xy = 1$, then $yx = 1$.

Let $\varphi: A \to A$
$a \mapsto ay$

$\varphi(x) = xy = 1 \Rightarrow \varphi$ surjective hom.

But A fin-dim'l $\mathbb{C}$-alg $\Rightarrow$ A noeth $\Rightarrow$ φ automorphism (by #4)

Now, WTS: $1 - yx = 0$

$$\varphi(1-yx) = (1-yx)y = y - yxy = y - y1 = y - y = 0$$

$\Rightarrow 1 - yx \in \ker\varphi$

But φ autom, hence $\ker\varphi = 0 \Rightarrow 1 - yx = 0 \Rightarrow yx = 1$

(7) A, B, C fin-gen modules over a PID R.
Show $B \cong C \Leftrightarrow A \oplus B \cong A \oplus C$.

($\Rightarrow$) $B \cong C \Rightarrow B$ & C have same inv. factors & free rk by Fund Thm of F.G. Mod / PID.

Therefore $A \oplus B$ & $A \oplus C$ also have same inv. factors & free rk, hence $A \oplus B \cong A \oplus C$, again by FTFGM/PID

($\Leftarrow$) $A \oplus B \cong A \oplus C \Rightarrow A \oplus B$ & $A \oplus C$ have same inv. factors & free rk,
i.e. for $A = R^m \oplus R/(a_1) \oplus \cdots \oplus R/(a_r)$ $\quad C = R^k \oplus R/(c_1) \oplus \cdots \oplus R/(c_t)$
$B = R^n \oplus R/(b_1) \oplus \cdots \oplus R/(b_s)$

$$A \oplus B \cong A \oplus C \Rightarrow R^{m+n} \oplus R/(a_1) \oplus \cdots \oplus R/(a_r) \oplus R/(b_1) \oplus \cdots \oplus R/(b_s)$$
$$\cong R^{m+k} \oplus R/(a_1) \oplus \cdots \oplus R/(a_r) \oplus R/(c_1) \oplus \cdots \oplus R/(c_t)$$

$\Rightarrow n = k$, $s = t$, and $\{b_1, \ldots, b_s\} = \{c_1, \ldots, c_t\}$

$\Rightarrow B \cong C$.