Algebra Qualifying Exam January 2004

(1) Determine the number of nonisomorphic groups of order $2 \cdot 7 \cdot 17 \cdot 23$.

(2) Let G be a finite p-group, p a prime. Prove that the following are equivalent:

(a) G does not contain a subgroup isomorphic to $\mathbb{Z}/p \times \mathbb{Z}/p$;

(b) Every abelian subgroup of G is cyclic;

(c) G has a unique subgroup of order p.

(3) Let K be a splitting field of $x^4 - 2$ over the rational numbers $\mathbb{Q}$.

(a) Find $[K : \mathbb{Q}]$ and describe the Galois group of $K/\mathbb{Q}$.

(b) How many intermediate fields are normal (Galois) over $\mathbb{Q}$? Explain.

(4) Let k be a commutative and let R, S be commutative k algebras such that R is noetherian and S is a finitely generated k-algebra. Prove that $R \otimes_k S$ is a noetherian ring.

(5) Let k be a field, B a finitely generated, k-algebra and let A be a k-subalgebra of B. commutative.

(a) If M is a maximal ideal of B, prove that $M \cap A$ is a maximal ideal of A.

(b) Give an example to show that this is false if B is not finitely generated.

(6) Let A be a 5 dimensional algebra over the field k of p elements, p a prime. Assume that for each nonzero $a \in A$, there exists $b \in A$ with $ab = e = e^2 \neq 0$. Find all such algebras up to isomorphism (note that this condition is satisfied by $M_n(F)$ for any field F and you may use this fact).

(7) Let F be a field and $F[x]$ the polynomial ring over F. Let M be a finitely generated free module over $F[x]$. Let $N_i, i = 1, 2, \ldots$ be a descending chain of $F[x]$-submodules of M. Prove that there exists a positive integer t so that for $i > t$, N_i/N_{i+1} is finite dimensional over F (note that $F[x]/I$ is finite dimensional over F for any nonzero ideal I).

a nilp. ⇒ $a^k = 0$ not nilp

∃ b s.t. $cb = e$

$0 = a^{n-1}ab = a^{n-1}e$

~~a zero d. ⇒ $a^n = 0$~~

~~∃ b s.t. $ab = e$~~

$\cancel{a^{n-1}ab} = a^{n-1}e$

$= 0$

$e = e^2$ ⇒ not nilpotent

for e, ∃ b s.t. $eb = e$

~~$ab = e$~~

⇒ $(ab)^k = e^k = e$

~~$(ab)^k = ab$~~

$ab = ab = ab$

$a \in A$, $\exists b \in A$ w/ $ab = e = e^2$

suppose $a^k = 0$; $\exists b$ s.t. $ab = e$

$\exists c$ s.t. $bc = e$

$\exists f$ s.t. $df = e$

$ab = e$

$a^{k-1}ab = \underbrace{a^{k-1}e}_{d} = ea^{k-1} = a^{k-1}eab$

$ab = e$; $a\,a^{k-1} = 0$

$\Rightarrow 0 = b0 = baa^{k-1} = a^{k-1}e = a^{k-1}ee$

$\Rightarrow a^{k-1}ab = a^{k-1}e$

$0 = a$

$0 = e$; $e^2 = e \Rightarrow e^2 - e = 0 \Rightarrow e(e-1) = 0$

know $e \neq 0$

~~for each~~

let $\{a_1, \ldots, a_5\}$ basis

Then $\exists b_i$ for each a_i s.t. $a_i b_i = e$

and $b_i = \sum_j \alpha_j^{(i)} a_j$

$e = a_1 b_1 = a_1 \sum_j \alpha_j^{(1)} a_j$

$= \sum_j \alpha_j^{(1)} a_1 a_j = \alpha_1 a_1^2 + \alpha_2 a_1 a_2 + \alpha_3 a_1 a_3 + \alpha_4 a_1 a_4 + \alpha_5 a_1 a_5$

① Groups of order $2\cdot7\cdot17\cdot23$

Sylow: $r_{23} \equiv 1 \pmod{23}$ and $r_{23} \mid 2\cdot7\cdot17 \Rightarrow r_{23} = 1, \cancel{2}, \cancel{7}, \cancel{2\cdot7}, \cancel{17}, \cancel{2\cdot17}, \cancel{7\cdot17}, \cancel{2\cdot7\cdot17}$
↳ normal

$r_{17} \equiv 1 \pmod{17}$ and $r_{17} \mid 2\cdot7\cdot23 \Rightarrow r_{17} = 1, \cancel{2}, \cancel{2\cdot7}, \cancel{2\cdot23}, \cancel{7\cdot23}, \cancel{7}, \cancel{23}, \cancel{2\cdot7\cdot23}$
↳ normal.

$r_{7} \equiv 1 \pmod{7}$ and $r_{7} \mid 2\cdot17\cdot23 \Rightarrow r_{7} = 1, \cancel{2}, \cancel{17}, \cancel{23}, \cancel{2\cdot17}, \cancel{2\cdot23}, \cancel{17\cdot23}, \cancel{2\cdot17\cdot23}$
↳ normal

Therefore, letting N, P, Q be the 23, 17, 7-Sylow subgroups respectively, we get NPQ is a normal subgroup.

Now consider S a Sylow 2-subgroup; we get $S \cap NPQ = 0$ since orders coprime; so $SNPQ = G$ and thus: get $G = NPQ \rtimes S$ and $\varphi: S \longrightarrow \mathrm{Aut}(NPQ)$
$s \longmapsto \sigma_s(n) = sns^{-1}$, hence $\sigma_s{}^2 = id$

Now N, P, Q coprime, prime orders, hence $NPQ \cong \mathbb{Z}_{23} \oplus \mathbb{Z}_{17} \oplus \mathbb{Z}_7$.
So $\mathrm{Aut}(\mathbb{Z}_{23} \oplus \mathbb{Z}_{17} \oplus \mathbb{Z}_7) \cong \mathbb{Z}_{22} \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_6$

case σ_s order 1: group is abelian and $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{23} \oplus \mathbb{Z}_{17} \oplus \mathbb{Z}_7$

case σ_s order 2: if $\mathbb{Z}_{22} \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_6 \cong \langle a \rangle \oplus \langle b \rangle \oplus \langle c \rangle$, then order 2 elts are

$$\left.\begin{matrix} \overset{\theta_1}{(a^{11}, 1, 1)} & \overset{\theta_2}{(1, b^8, 1)} & \overset{\theta_3}{(1, 1, c^3)} \\ (a^{11}, b^8, 1) & (1, b^8, c^3) & \cancel{(a^{11}, 1, c^3)} \\ (a^{11}, 1, c^3) & (a^{11}, b^8, c^3) & \end{matrix}\right\} \text{7 elts}$$

$\rightarrow \theta_1, \theta_2, \theta_3, \text{“}\theta_1 + \theta_2\text{”}, \text{“}\theta_1 + \theta_3\text{”}, \text{“}\theta_2 + \theta_3\text{”}, \text{“}\theta_1 + \theta_2\text{”}; \text{“}\theta_1 + \theta_2 + \theta_3\text{”}.$

let $\mathbb{Z}_{23} \oplus \mathbb{Z}_{17} \oplus \mathbb{Z}_7 \cong \langle \alpha \rangle \oplus \langle \beta \rangle \oplus \langle \gamma \rangle$

want m, l, k s.t.

$$m^2 \equiv 1 \pmod{23}$$
$$l^2 \equiv 1 \pmod{17}$$
$$k^2 \equiv 1 \pmod{7}$$

$m^2 - 1 \equiv 0 \Rightarrow (m-1)(m+1) \equiv 0$

- $\Rightarrow (m-1)(m+1) \equiv 0 \pmod{23}$
  $m = 22$
- $(l+1)(l-1) \equiv 0 \pmod{17}$
  $\rightarrow l = 16$
- $(k-1)(k+1) \equiv 0 \pmod{7}$
  $\Rightarrow k = 6$

$\alpha^{23} = \beta^{17} = \gamma^{7} = 1$

$\theta_1(\alpha, \beta, \gamma) = (\alpha^{22}, \beta, \gamma)$
$\theta_2(\alpha, \beta, \gamma) = (\alpha, \beta^{16}, \gamma)$ and linear combinations thereof, so:
$\theta_3(\alpha, \beta, \gamma) = (\alpha, \beta, \gamma^{6})$

α $G = \langle s^2 = \alpha^{23} = \beta^{17} = \gamma^7 = 1 \mid s\alpha s^{-1} = \alpha^{22} \quad s\beta s^{-1} = \beta \quad s\gamma s^{-1} = \gamma \rangle$
$\alpha\ \beta$ $G = \langle s^2 = \alpha^{23} = \beta^{17} = \gamma^7 = 1 \mid s\alpha s^{-1} = \alpha^{22} \quad s\beta s^{-1} = \beta^{16} \quad s\gamma s^{-1} = \gamma \rangle$
$\alpha\ \gamma$ $G = \langle s^2 = \alpha^{23} = \beta^{17} = \gamma^7 = 1 \mid s\alpha s^{-1} = \alpha^{22} \quad s\beta s^{-1} = \beta \quad s\gamma s^{-1} = \gamma^6 \rangle$
γ $G = \langle s^2 = \alpha^{23} = \beta^{17} = \gamma^7 = 1 \mid s\alpha s^{-1} = \alpha \quad s\beta s^{-1} = \beta \quad s\gamma s^{-1} = \gamma^6 \rangle$
$\beta\ \gamma$ $G = \langle s^2 = \alpha^{23} = \beta^{17} = \gamma^7 = 1 \mid s\alpha s^{-1} = \alpha \quad s\beta s^{-1} = \beta^{16} \quad s\gamma s^{-1} = \gamma^6 \rangle$
β $G = \langle s^2 = \alpha^{23} = \beta^{17} = \gamma^7 = 1 \mid s\alpha s^{-1} = \alpha \quad s\beta s^{-1} = \beta^{16} \quad s\gamma s^{-1} = \gamma \rangle$
$G = \langle s^2 = \alpha^{23} = \beta^{17} = \gamma^7 = 1 \mid s\alpha s^{-1} = \alpha^{22} \quad s\alpha s^{-1} = \beta^{16} \quad s\alpha s^{-1} = \gamma^6 \rangle$

All distinct since
cannot obtain generators of one
from the others (different centers)

② $|G| = p^k$ (finite p-group).
Prove TFAE: (a) $\nexists$ sg. $\mathbb{Z}_p \oplus \mathbb{Z}_p \leq G$
(b) Every abelian sg. of G is cyclic (c) G has a unique sg. of order p.

<u>(a) ⇒ (b)</u>: Suppose $\nexists$ sg. $\mathbb{Z}_p \oplus \mathbb{Z}_p$ of G, hence there is no subgroup $\bigoplus_{i=1}^{n} \mathbb{Z}_p$ of G for any n. Now, an abelian subgroup of G must be of the form $\mathbb{Z}_{p^{e_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{e_l}}$, but this has a sg. $\mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p$, hence $l = 1$, hence abelian sg. $\mathbb{Z}_{p^e}$, hence cyclic.

<u>(b) ⇒ (a)</u>: Every abelian subgroup of G is cyclic, so $\mathbb{Z}_p \oplus \mathbb{Z}_p \not\leq G$ since $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is not cyclic

<u>(b) ⇒ (c)</u>: Suppose every abelian sg. of G is cyclic.
Recall that a p-group has a nontrivial center $Z(G)$
Suppose that $|Z(G)| = p$ and $|P| = p$ with $P \cap Z(G) = 1$.
$Z(G)$ is normal, hence $Z(G)P$ a subgroup of order p^2 in G.
So it is abelian, hence cyclic, hence its subgroup of order p must be unique, hence $P = Z(G)$.

Suppose $|Z(G)| = p^k$; it is an abelian subgroup, hence cyclic, hence has unique sg. order P of order p. Suppose there is another sg. P' of order p. It is not central, hence nonabelian, contradiction since order p. So order p sg. is unique.

<u>(c) ⇒ (a)</u>: G has a unique sg. of order p. If there was sg. $\mathbb{Z}_p \oplus \mathbb{Z}_p \leq G$, then there are 2 sg's order p, contradiction.
So there is no sg. $\mathbb{Z}_p \oplus \mathbb{Z}_p$

③ $K/\mathbb{Q}$ splitting field of x^4-2.

(a) Find $[K:\mathbb{Q}]$ and describe $\mathrm{Gal}(K/\mathbb{Q})$:

In a splitting field, $x^4-2=(x^2-\sqrt{2})(x^2+\sqrt{2})$

$$=(x-\sqrt[4]{2})(x+\sqrt[4]{2})(x-i\sqrt[4]{2})(x+i\sqrt[4]{2})$$

So the roots are $i^k\sqrt[4]{2}$, $k=1,\dots,4$, so $\mathbb{Q}(i,\sqrt[4]{2})$ a splitting field.

Now consider the tower:

$$\begin{array}{ll}
K=\mathbb{Q}(i,\sqrt[4]{2}) & \\
\quad\Big| \;\Big)\, 4 & \leftarrow \text{since } x^4-2 \text{ irred. over } \mathbb{Q}(i) \\
\mathbb{Q}(i) & \quad\;\; \text{since } \sqrt{2},\sqrt[4]{2}\notin\mathbb{Q}(i) \\
\quad\Big| \;\Big)\, 2 & \\
\mathbb{Q} &
\end{array}$$

So $[K:\mathbb{Q}]=8$ by tower law.

Now consider the elements of $\mathrm{Gal}(K/\mathbb{Q})$:

$$\sigma: \sqrt[4]{2}\mapsto i\sqrt[4]{2},\quad i\mapsto i \qquad\qquad \tau: \sqrt[4]{2}\mapsto\sqrt[4]{2},\quad i\mapsto -i \quad \text{complex conj.}$$

$$\rightsquigarrow \sigma^4=\mathrm{id} \qquad\qquad \rightsquigarrow \tau^2=\mathrm{id}.$$

and see that

$$\tau\sigma\tau(\sqrt[4]{2})=\tau\sigma(\sqrt[4]{2})=\tau(i\sqrt[4]{2})=-i\sqrt[4]{2}$$
$$=\sigma^3(\sqrt[4]{2})=\sigma^{-1}(\sqrt[4]{2})$$

$$\tau\sigma\tau(i)=\tau\sigma(-i)=\tau(-i)=i=\sigma^{-1}(i)$$

hence $\tau\sigma\tau=\sigma^{-1}$, and

$$\mathrm{Gal}(K/\mathbb{Q})\cong\langle\tau,\sigma;\ \tau^2=\sigma^4=1,\ \tau\sigma\tau=\sigma^{-1}\rangle$$
$$\cong D_8.$$

④ k commutative, R, S comm k-algebras s.t. R noeth, S fin-gen

Prove $R \otimes_k S$ is noeth:

Since S is finitely-generated k-algebra and $k \subseteq R$ since R k-alg., $R \otimes_k S$ is a finitely-generated R-algebra.

(i.e., if $\{s_1, \dots, s_n\}$ k-generating set for S, then $\{1 \otimes_k s_i\}_{i=1}^n$ is an R-generating set for $R \otimes_k S$)

Now, R is noeth and $R \otimes_k S$ is a finitely gen R-alg, so there is surjective hom

$$f : R[x_1, \dots, x_m] \longrightarrow R \otimes_k S$$

But R noeth $\Rightarrow$ $R[x_1, \dots, x_m]$ noeth by Hilbert basis thm,

hence $R \otimes_k S = f(R[x_1, \dots, x_n])$ also noeth since f a homomorphism.

(5) k field, B fin-gen commutative k-algebra, and A a k-subalgebra of B.

(a) If M maximal ideal of B, prove $M \cap A$ max ideal in A ?

B/M a field; now consider the map:

$$\begin{aligned} \varphi : A &\hookrightarrow B \longrightarrow B/M \\ x &\mapsto x \longmapsto a+M \end{aligned}$$

hence $\ker\varphi = A \cap M$, hence $A/M\cap A \to B/M$ is an injective map; specifically, $A/M\cap A$ is injected into a field, hence an integral domain.

Now, B/M is a finite-gen extension of k,

$$\begin{tikzcd} B/M \arrow[d, dash] \arrow[dr, dash] & \\ k \arrow[r, dash] & A/M\cap A \end{tikzcd}$$

⑥ A is a k-algebra with $\dim_k A = 5$, $|k| = p$.
Assume for each $0 \neq a \in A$, $\exists b \in A$ with $ab = e = e^2 \neq 0$. Find all such algebras up to isomorphism

A fin-dim'l k-algebra $\Rightarrow$ A artinian $\Rightarrow$ $J(A)$ nilpotent
Now, suppose $a \in A$ w/ $a^n = 0$ some minimal n; see that since $e^n = e$ for all e, $e \notin J(A)$ (not nilpotent).
So choose $x \in J(A)$; then $\exists\, b \in A$ s.t. $bx = e \notin J(A)$; but $J(A)$ is an ideal, hence $x = 0$, here $J(A) = 0$.
Therefore A is artinian + J.S.S. $\Rightarrow$ S.S.
Apply Artin-Wedderburn.
Then $A \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$.
But D_i's are finite dimensional algebras over finite fields, hence finite, hence fields by little Wedderburn.
These are finite field extensions of $\mathbb{F}_p$, hence $D_i = \mathbb{F}_{p^{m_i}}$

So A can be:

- $\mathbb{F}_p \oplus \mathbb{F}_p \oplus \mathbb{F}_p \oplus \mathbb{F}_p \oplus \mathbb{F}_p$
- $\mathbb{F}_{p^2} \oplus \mathbb{F}_{p^3}$
- $\mathbb{F}_{p^2} \oplus \mathbb{F}_{p^2} \oplus \mathbb{F}_p$
- $\mathbb{F}_{p^2} \oplus \mathbb{F}_p \oplus \mathbb{F}_p \oplus \mathbb{F}_p$
- $\mathbb{F}_p \oplus \mathbb{F}_p \oplus \mathbb{F}_{p^3}$
- $M_2(\mathbb{F}_p) \oplus \mathbb{F}_p$
- $\mathbb{F}_{p^5}$
- $\mathbb{F}_{p^4} \oplus \mathbb{F}_p$

(7) F field, F[x] poly ring.

- let M be fin-gn free module over $F[x]$.
- let N_i, $i=1,2,\ldots$ be descending chain of $F[x]$-submodules of M

prove $\exists$ positive integer t s.t., for $i > t$, N_i/N_{i+1} is fin-dim over F. (note: $F[x]/I$ fin-dim over F for any nonzero ideal)

M is a finitely-generated module over a PID, hence

$M \cong (F[x])^k$

An $F[x]$ submodule will also be free since fin-gen mod/PID

$N_1 \supseteq N_2 \supseteq N_3 \supseteq \cdots N_t \supseteq N_{t+1}$ want N_t/N_{t+1} fin-dim over F.

~~after $i > t$~~

~~for~~ since the descending chain is stuck, ~~after~~ for $i > k$, N_i must be a direct sum of ideals of $F[x]$

then M/N_{i+1} is finite dim, hence N_i/N_{i+1} is