ALGEBRA EXAM SEPTEMBER 2004

Do as many problems as you can

1. Up to isomorphism describe all groups of order $399 = 3 \cdot 7 \cdot 19$. For each group find the order of its center and the order of its commutator subgroup.

~~2.~~ Suppose R is a finite dimensional algebra over a field F with 1 and $U(R)$,the group of units in R, is abelian. Show that the Jacobson radical $J(R)$ and $R/J(R)$ are commutative.

~~3.~~ Let L be a subfield of the finite field K of characteristic p. Let $\alpha \in K$ with minimal polynomial $v(x)$ of degree d over L. Show that $v(x)$ splits over K and that for some $q = p^m$ the roots of $v(x)$ in K are $\{\alpha, \alpha^q, \ldots, \alpha^{q^{d-1}}\}$.

→ 4. Let R be a commutative ring with 1 and M, N, V all R-modules.

→ (a) If M and N are projective show that $M \otimes_R N$ is also a projective R-module.

(b) Let

$$\mathrm{Tr}(V) = \{\sum_{i=1}^{n} \phi_i(v_i) | \phi_i \in \mathrm{Hom}_R(V, R), v_i \in V, n = 1, 2, \ldots\}.$$

If $1 \in \mathrm{Tr}(V)$ show that up to isomorphism some finite direct sum V^k contains R as an R-module direct summand.

~~5.~~ Show that any surjective ring homomorphism $f : R \to R$ of a left Noetherian ring R must be an isomorphism. Give an example to show this may be false if the ring is not noetherian. infinite-variable polynomial ring

~~6.~~ In $\mathbb{C}[x, y]$ show that some power of $(x + y)(x^2 + y^4 - 2)$ is in the ideal $(x^3 + y^2, y^3 + xy)$.

$K = \text{at } \mathbb{F}_{p^{km}}$ α not of $x^{p^{km}} - x$ ($\mathbb{F}_{p^{km}}$ spl. field over $\mathbb{F}_p$)

$L = \mathbb{F}_{p^m}$ so $\Rightarrow v(x) \mid x^{p^{km}} - x$ hence v splits into linear factors over K

$\mathbb{F}_p$

~~$\Rightarrow u(x)v(x) = x^{p^{km}} - x$~~

~~But $u(x) \neq 0$, hence~~

$v(x) \mid x^{p^{km}} - x$

② R finite-dimensional F-algebra, $U(R)$ units is abelian.
Show that $J(R)$ and $R/J(R)$ commutative

- R fin-dim'l F-algebra $\Rightarrow$ artinian $\Rightarrow$ $J(R)$ nilpotent
Therefore, for $x, y \in J(R)$, $1-x, 1-y \in U(R)$, hence
$(1-x)(1-y) = (1-y)(1-x) \Rightarrow 1-x-y+xy = 1-y-x+yx$
$\Rightarrow xy = yx \Rightarrow$ $J(R)$ is commutative

- Consider $R/J(R)$; it is artinian & J s.s., hence s.s., so apply Artin-Wedderburn: $R/J(R) \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$.
Then since $U(R)$ is commutative & $U(R) \cap J(R) = 0$, $U(R/J(R))$ is also commutative, but also recall that
$U(R/J(R)) = U(M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)) = GL_{n_1}(D_1) \oplus \cdots \oplus GL_{n_k}(D_k)$,
hence $n_i = 1\ \forall i$, $D_i = F_i$, field, for all i, since $U(R/J(R))$ comm.
Therefore $R/J(R) \cong F_1 \oplus \cdots \oplus F_k$, so commutative.

(3) K/L, $\alpha \in K$ has min poly $v(x) \in L[x]$, $\deg v(x) = d$.

(1) <u>$v(x)$ splits in K</u>:

$$\begin{array}{c} K = \mathbb{F}_{p^{km}} \\ | \\ L = \mathbb{F}_{p^m} \\ | \\ \mathbb{F}_p \end{array}$$

$K = \mathbb{F}_{p^{km}}$; $\alpha \in K$, hence α a root of $x^{p^{km}} - x$, hence $v(x) \mid x^{p^{km}} - x$, hence $v(x)$ also splits into linear factors in K.

Now $v(x)$ splits, hence $[L(\alpha) : L] = d$

hence if $L = \mathbb{F}_{p^m} = \mathbb{F}_q$, then $L(\alpha) = \mathbb{F}_{q^d} \subseteq K$

Since $v(x)$ is irreducible over L, the Galois group $G(L(\alpha)/L)$ acts transitively on the roots, hence the set of roots is $\{\sigma(\alpha)\}_{\sigma \in \mathrm{Gal}(L(\alpha)/L)}$

Now, $\sigma : \alpha \mapsto \alpha^q$ generates the Galois group, hence $\{\alpha, \alpha^q, \alpha^{q^2}, \dots, \alpha^{q^{d-1}}\}$ are the roots of $v(x)$.

Clearly $L(\alpha) \subseteq K$ is a subfield, hence these are the roots in K

④ R commutative ring w/ 1, M, N, V R-modules

(a) If M, N projective, $M \otimes_R N$ is also projective

~~M, N projective.~~

~~Cor 32: $Hom_R(N, -)$ exact~~

~~$0 \to A \to B \to M \otimes N \to 0$~~ $\quad 0 \to A \to B \to M \to 0$

~~$0 \to A \to B \to N \to 0$~~

~~$0 \to M \otimes A \to M \otimes B \to M \otimes N \to 0$~~

Adjoint associativity:

$H_R(A \otimes_R B, C) \cong Hom_R(A, Hom_R(B, C))$ $\quad$ R commutative?

M, N projective

$\to H_R(M, \_)$, $H_R(N, \_)$ exact $\Rightarrow H_R(M, H_R(N, \_))$ exact

$$\parallel$$

$H_R(M \otimes_R N, \_)$ exact

$\Rightarrow$ $M \otimes N$ projective

$L(\alpha)$

$(\alpha + \alpha^q + \alpha^{q^2} + \cdots + \alpha^{q^{d-1}})$

$d \mid [K:L]$ and $m \mid [K:L]$

(b) $Tr(V) = \{ \sum_{i=1}^{n} \phi_i(v_i) \; : \; \phi_i \in Hom_R(V,R),\ v_i \in V,\ n = 1, 2, \ldots \}$

If $Tr(V) \ni 1$, show some direct sum V^k contains R as an R-mod direct summand.

⑤ Any surjective ring hom $f: R \to R$ of left noeth rings must be isomorphism

- $f: R \to R$ surj, so f^k surj.
  $\ker f \subseteq \ker f^2 \subseteq \dots \subseteq \ker f^n = \ker f^{n+1}$ some n; now choose $x \in \ker \varphi$.
  Then $\exists y$ such that $f^n(y) = x \Rightarrow f^{n+1}(y) = f(x) = 0 \Rightarrow y \in \ker f^{n+1}$
  $= \ker f^n$
  so then $x = f^n(y) = 0 \Rightarrow \ker \varphi = 0$

- example to show not true if not noeth.

⑥ ① $\mathbb{C}[x,y]$; show $(x+y)(x^2+y^4-2) \in \sqrt{(x^3+y^2, y^3+xy)}$

$$\sqrt{(x^3+y^2, y^3+xy)} = Id(Var(x^3+y^2) \cap Var(y^3+xy))$$

$$\cancel{= \sqrt{Id(Var(x^3+y^2)) \cap Id(Var(x^3+y^2))}}$$

see that $(0,0) \in Var(x^3+y^2) \cap Var(y^3+xy)$,

hence $Id(Var(\cdot) \cap Var(\cdot)) \subseteq (x,y)$

$\cancel{(x+y)(x^2+y^4-2)}$

$\cancel{x^3+xy^4-2x+yx^2+y^5-2y}$

$x=-y$

$\cancel{x=-1}$

$\cancel{\Rightarrow y=1}$

$\cancel{\Rightarrow 1+(-1)(1)=0}$ ✓

$\cancel{x^3+y^2=0}$

$\cancel{\Rightarrow x^3=-y^2}$

$\cancel{\Rightarrow x=y=1}$

$\cancel{y^3+xy=0}$

$\cancel{\Rightarrow y^3=-xy}$

$\cancel{\Rightarrow y^2=-x}$

$Var(x^3+y^2, y^3+xy)$

$x^3=-y^2$

$y^3=-xy \Rightarrow \cancel{y}(-x^3) = -x\cancel{y}$

$\Rightarrow x^2=1 \Rightarrow x=\pm 1$

But $x^3=-y^2$

hence $-(1)^3=y^2 \Rightarrow -1=y^2 \Rightarrow y=i$

$-(-1)^3=y^2 \Rightarrow 1=y^2 \Rightarrow y=1$

$$\left\{\begin{matrix}(1,i)\\(-1,1)\end{matrix}\right\} = Var(x^3+y^2, y^3+xy)$$

and see that

- $(x,y)=(-1,1)$: $(-1+1)(x^2+y^4-2)=0$ ✓
- $(x,y)=(1,i)$: $(x+y)(\underbrace{1+1-2}_{0})=0$ ✓

hence $(x+y)(x^2+y^4-2) \in Id(Var(x^3+y^2, y^3+xy))$

$= \sqrt{(x^3+y^2, y^3+xy)}$