510ab

$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \cong \langle \alpha \rangle \oplus \langle \beta \rangle$

$\theta(\alpha,\beta) = (\alpha^2, \beta)$

(α, β^2)

(α^2, β^2)

Algebra Qualifying Exam February 2003

Partial credit is given for partial solutions

DBF p. 136-137

1. Show that there are ~~four~~ five non-isomorphic groups of order 18. For partial credit show that there are at most four non-isomorphic groups of order 18. group classification

2. Let D_n be the dihedral group of order $2n$. Recall that D_n is generated by an element a of order n, an element b of order 2, and the relation $bab = a^{-1}$ holds.
i) Show that the derived group D_n' is:

$$D_n' = \begin{cases} \mathbb{Z}/n\mathbb{Z} & \text{for } n \text{ odd;} \\ 2\mathbb{Z}/n\mathbb{Z} & \text{for } n \text{ even.} \end{cases}$$

(Hint: Show first that $a^2 \in D_n'$.) Conclude that D_n is solvable.
ii) Show that: D_n is nilpotent if and only if n is a power of 2. (Hint: Show that for $i > 1$, $\gamma_i(D_n) = \langle a^{2^{i-1}} \rangle$, where γ_i is the lower central series.)

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = I$

use resolvant cubic, discriminant, etc.

3. Let $f(X) \in K[X]$ be an irreducible polynomial of degree 4 and G its Galois group. Show that G is isomorphic to one of the following groups: $\mathbb{Z}/4$, $\mathbb{Z}/2 \times \mathbb{Z}/2$, D_4, A_4, or S_4.

4. Let $f(X) = (X^3 - 2)(X^2 - 3) \in \mathbb{Q}[X]$. Find the splitting field K of f over $\mathbb{Q}$, the Galois group of the extension $\mathbb{Q} \subseteq K$ and give the number of subfields of K of each degree. Galois group

5. i) How many different similarity classes of 4×4 nilpotent matrices are there over a field F?
ii) What is the characteristic polynomial of such a matrix?
iii) Prove or give a counterexample: If two such matrices have the same minimum polynomial and characteristic polynomial, they are similar.

6. Let $R = \mathbb{Q}[x, y, z]$.
i) Prove that every simple R-module M is finite dimensional over $\mathbb{Q}$.
ii) Show that there is no bound on the dimension of a simple module.

generalized Nullstellensatz?

7. Consider a homogeneous system of m linear equations in n variables over $\mathbb{Z}$ and assume that $d = n - m \geq 0$.
i) Show that the set of solutions is a free abelian group of rank at least d.
ii) What is the rank of the group of solutions (in terms of the coefficient matrix of the system of equations)?

$ab - cd \neq 0$ $\quad ab \neq cd$

$\alpha^3 = 1$

$\begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ 0 & \gamma \end{bmatrix} = \begin{bmatrix} \alpha^2 & \alpha\beta + \beta\gamma \\ 0 & \gamma^2 \end{bmatrix}$

$= \begin{bmatrix} \alpha^3 & \alpha^2\beta + \gamma(\alpha\beta + \beta\gamma) \\ 0 & \gamma^3 \end{bmatrix}$

$\alpha \in \mathbb{F}_3 \to \alpha^3 = \alpha \Rightarrow \alpha^2 = 1$

$\alpha \in \mathbb{F}_3^\times = \mathbb{Z}_2$

$\alpha, \gamma = 1 \quad \begin{bmatrix} 1 & 3\beta \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix}1&b\\0&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1+b\\1\end{bmatrix}$

$\begin{bmatrix}1&0\\c&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\c+1\end{bmatrix}$

$\begin{bmatrix}0&a\\a&0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}a\\a\end{bmatrix}$

$\begin{bmatrix}a&b\\c&d\end{bmatrix}^2=I$

$\begin{bmatrix}a&b\\c&d\end{bmatrix}^{-1}=\dfrac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}=\begin{bmatrix}a&b\\c&d\end{bmatrix}$

$\begin{bmatrix}a&b\\c&d\end{bmatrix}^2=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}a^2+bc&ab+bd\\ca+dc&cb+d^2\end{bmatrix}$

$b(a+d)=ab+bd=0$

$a^2+bc=1$

$cb+d^2=1$

$c(a+d)=ca+dc=0$

$\cancel{1,\alpha,\alpha^2}$

$1,\alpha,\alpha^2$

$r_2\equiv 1 \pmod 2$ and $r_2 \mid 3$ $\Rightarrow r_2=1,3$

$bc=1-a^2=(1-a)(1+a)$

$bc=1-d^2=(1-d)(1+d)$

$a=1 \iff d=1$

~~$a=2 \iff d=2$~~

$b(a+d)=0$

$c(a+d)=0$

$\mathbb{Z}_2\oplus\mathbb{Z}_2$

$\mathbb{Z}_2\oplus\mathbb{Z}_2$

$G/C_G(P)\le \mathrm{Aut}(P)$

$a,d=0$

$\Rightarrow bc=1$

$\Rightarrow b=c^{-1}$

$\Rightarrow$

$\Downarrow$

$bc=0$

$\Downarrow$

$b=0, c\neq 0 \qquad c=0, b\neq 0$

$\Downarrow$

$(a+d)=0$

$\Downarrow$

$2=a=-d$

$\Rightarrow d=1$ ↯, so

$bc(a+d)^2=$

$x^2-I=0$

$\Rightarrow (X-I)(X+I)=0$

$\begin{bmatrix}0&a\\a&0\end{bmatrix}\begin{bmatrix}0&a\\a&0\end{bmatrix}=\begin{bmatrix}a^2&0\\0&a^2\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

$\underset{2}{\begin{bmatrix}1&b\\0&1\end{bmatrix}},\ \underset{2}{\begin{bmatrix}1&0\\c&1\end{bmatrix}},\ \underset{2}{\begin{bmatrix}0&a\\a&0\end{bmatrix}}$

① $|G| = 18 = 3^2 \cdot 2$

$r_3 \equiv 1 \pmod 3$ and $r_3 | 2 \Rightarrow r_3 = 1 \rightarrow N$ the Sylow 3-s.g. is normal.

choose S a Sylow 2-s.g. and then since orders are coprime, $NS \cong G$.

N is a p^2 order group, hence cyclic or elem-abelian, i.e. $N \cong \mathbb{Z}_9$ or $N \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$. (let $S = \langle s \rangle$)

Then consider $\varphi : S \longrightarrow Aut(N)$; $\varphi(s)$ must be order 1 or 2

Case 1: $N \cong \mathbb{Z}_9$, hence $Aut(N) \cong (\mathbb{Z}_9)^\times \cong \mathbb{Z}_{3(3-1)} \cong \mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$.

$\varphi(s)$ must be order 1 or 2, and there is one of each, in $\mathbb{Z}_2 \oplus \mathbb{Z}_3$. Note that if order 1, abelian case $\Rightarrow$ $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_9$. ①

Consider order 2; let $\mathbb{Z}_9 \cong \langle \alpha \rangle$ and consider $\theta \in Aut(\mathbb{Z}_9)$ with $\theta^2 = id$; $\theta(\alpha) = \alpha^{k^2}$ such that $k^2 \equiv 1 \pmod 9 \Rightarrow (k-1)(k+1) \equiv 0 \pmod 9$

$\Rightarrow k = 8$, hence $\varphi(s)(\alpha) = \theta(\alpha) = \alpha^8 \Rightarrow \alpha^8 = s\alpha s^{-1}$

$\Rightarrow G \cong \langle s, \alpha : s^2 = \alpha^9 = 1, s\alpha s^{-1} = \alpha^8 \rangle$ ②

Case 2: $N \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$, hence $Aut(N) \cong GL_2(\mathbb{Z}_3)$, order $3 \cdot 2^4$

again $\varphi(s)$ order 1 or 2; order 1 is abelian case $\Rightarrow$ $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ ③

Now we'll find the order 2 elts in $Aut(N)$.

let $a \in GL_2(\mathbb{F}_3)$ be ord 2. ① cont'd.

$GL_2(\mathbb{F}_3)$; order 2 elts satisfy X^2-1, hence $m_a(X) \mid X^2-1$

$\{0,1,-1\}$ $\quad (X-1)(X+1)$,

hence

① $m_a(X) = (X-1)$

② $m_a(X) = (X+1)$

③ $m_a(X) = (X-1)(X+1)$

case ①: ~~eigenvalues both ①~~

$\cancel{a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2}$

" ② : — " — ⊖1

$\theta_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

" ③ eigenvalues 1 and −1 :

$\overset{a}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}, \overset{b}{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}, \overset{c}{\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}}, \overset{d}{\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}}$

conj.

these rep'n sim. classes by Jordan Canonical Form.

a, d } different similarity classes.
b, c

θ_2 $\quad \theta_3$

$\underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}}_{c} \underbrace{\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}}_{d} \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}}_{c^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}_{b}$

let $\mathbb{Z}_3 \oplus \mathbb{Z}_3 = \langle\alpha\rangle \oplus \langle\beta\rangle$

suppose $\theta_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$; $\theta_1(\alpha, \beta) = (\alpha^2, \beta^2)$

$\theta_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; $\theta_2(\alpha, \beta) = (\alpha, \beta^2)$

$\theta_3 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$; $\theta_3(\alpha, \beta) = (\alpha^2, \beta)$.

So $G = \langle s, \alpha, \beta : s^2 = \alpha^3 = \beta^3,\ s\alpha s^{-1} = \alpha^2,\ s\alpha s^{-1} = \beta^2 \rangle$

$G = \langle s, \alpha, \beta : s^2 = \alpha^3 = \beta^3,\ s\alpha s^{-1} = \alpha,\ s\alpha s^{-1} = \beta^2$

$G = \langle s, \alpha, \beta : s^2 = \alpha^3 = \beta^3\ \ s\alpha s^{-1} = \alpha^2,\ s\alpha s^{-1} = \beta \rangle$

① work

$\mathrm{Aut}(\mathbb{Z}_3 \oplus \mathbb{Z}_3) = GL_2(\mathbb{F}_3)$

Let $L \in GL_2(\mathbb{F}_3)$ and then $L: \mathbb{F}_3 \oplus \mathbb{F}_3 \longrightarrow \mathbb{F}_3 \oplus \mathbb{F}_3$.

basis' = $\{(1,0), (0,1)\}$

$\mathbb{Z}_3 = \langle \alpha \rangle \cong (1,0)$, $(0,1) \cong \langle \beta \rangle = \mathbb{Z}_3$

want order 2 elts:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{matrix} a^2+bc = 1 & b(a+d) = 0 \\ d^2+bc = 1 & c(a+d) = 0 \end{matrix}$$

$$\left.\begin{matrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b \\ d \end{bmatrix} \end{matrix}\right\} \text{must be lin. independent}$$

i.e. $x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix} = 0 \Rightarrow x = y = 0$

$xa + yb = 0$
$xc + yd = 0$

② D_n dihedral grp of order $2n$

Recall $D_n = \langle a, b : a^n = b^2 = 1,\ bab = a^{-1} \rangle$

(i) Show that $D_n' := [D_n, D_n] = \begin{cases} \mathbb{Z}/n\mathbb{Z}, & n \text{ odd} \\ 2\mathbb{Z}/n\mathbb{Z}, & n \text{ even} \end{cases}$ and conclude that D_n solvable.

(Hint: show first that $a^2 \in D_n'$)

Note that $D_n/\langle a \rangle \cong \mathbb{Z}_2$ is abelian, hence $D_n' \subseteq \langle a \rangle \cong \mathbb{Z}_n$ $(*)$

case n odd: $a^n = 1 \Rightarrow a^{2k}a = 1 \Rightarrow (a^2)^k = a^{-1}$

Now, $bab = a^{-1} \Rightarrow ba = a^{-1}b^{-1}$, hence $D_n' \ni aba^{-1}b^{-1} = abba = a^2$,

so $a^2 \in D_n' \Rightarrow (a^2)^k \in D_n' \Rightarrow a^{-1} \in D_n' \Rightarrow a \in D_n'$, hence D_n' has an order n element, hence $\mathbb{Z}_n \subseteq D_n'$, hence $\mathbb{Z}_n = D_n'$ by $(*)$.

case n even: In previous case, we showed $a^2 \in D_n'$ and a^2 is order $\frac{n}{2}$, hence D_n' has cyclic group of order $\frac{n}{2}$, i.e. $2\mathbb{Z}/n\mathbb{Z} \subseteq D_n'$ (since $D_n' \subseteq \mathbb{Z}_n$)

Now see that $D_n/\langle a^2 \rangle$ has $a^2 = b^2 = 1$ and $bab = a^{-1} \Rightarrow ab = b^{-1}a^{-1} = ba$,

hence abelian, so $D_n' \subseteq \langle a^2 \rangle \cong 2\mathbb{Z}/n\mathbb{Z}$, hence $D_n' = 2\mathbb{Z}/n\mathbb{Z}$.

Now D_n' is abelian, hence $D_n^2 = \{e\}$, hence derived series terminates $\Rightarrow$ solvable.

(ii) Show that D_n nilpotent $\Leftrightarrow$ n a power of 2

(Hint: show that for $i > 1$, $\gamma_i(D_n) = \langle a^{2^{i-1}} \rangle$ where γ_i is lower central series)

NOT COVERED

④ $f(x) = (x^3-2)(x^2-3) \in \mathbb{Q}[x]$

Find the splitting field $K/\mathbb{Q}$ & $\mathrm{Gal}(K/\mathbb{Q})$

Let ω be primitive 3rd root of unity. Then the roots of f are $\sqrt{3}, -\sqrt{3}, \sqrt[3]{2}, \omega\sqrt[3]{2}, \omega^2\sqrt[3]{2}$, hence a splitting field is $\mathbb{Q}(\sqrt{3}, \sqrt[3]{2}, \omega)$. Now consider the tower:

$$
\begin{array}{ll}
K = \mathbb{Q}(\sqrt{3}, \sqrt[3]{2}, \omega) & \\
\quad | & \Big\} 2 \quad \text{since min poly of } \sqrt{3} \text{ is still } x^2-3 \text{ over } \mathbb{Q}(\sqrt[3]{2},\omega); \\
\mathbb{Q}(\sqrt[3]{2}, \omega) & \\
\quad | & \Big\} 3 \quad \text{since min poly of } \sqrt[3]{2} \text{ over } \mathbb{Q}(\omega) \text{ is still } x^3-2. \\
\mathbb{Q}(\omega) & \\
\quad | & \Big\} 2 \quad \text{since cyclotomic} \\
\mathbb{Q} &
\end{array}
$$

note that even though $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$, clearly $\sqrt{3}$ cannot be obtained from lin. combination of ω & ω^2

So by tower law, $12 = [K:\mathbb{Q}] = |\mathrm{Gal}(K/\mathbb{Q})|$

Now consider the tower:

$$
\begin{tikzcd}
 & K = \mathbb{Q}(\sqrt{3}, \sqrt[3]{2}, \omega) \arrow[dl, dash, "6"'] \arrow[dr, dash, "2"] & \\
\mathbb{Q}(\sqrt{3}) \arrow[dr, dash, "2"'] & & \mathbb{Q}(\sqrt[3]{2}, \omega) \arrow[dl, dash, "6"] \\
 & \mathbb{Q} &
\end{tikzcd}
$$

these are Galois extensions, so $\mathrm{Gal}(K/\mathbb{Q}(\sqrt{3})) \trianglelefteq G$ index 2, $\mathrm{Gal}(K/\mathbb{Q}(\sqrt[3]{2},\omega)) \trianglelefteq G$ index 6.

$\sigma: \sqrt[3]{2} \mapsto \omega\sqrt[3]{2}$, $\omega \mapsto \omega$ order 3

$\tau: \sqrt[3]{2} \mapsto \sqrt[3]{2}$, $\omega \mapsto \omega^2$ order 2

(σ, τ fix $\mathbb{Q}(\sqrt{3})$)

generate $\mathrm{Gal}(K/\mathbb{Q}(\sqrt{3}))$ and have:

$\sigma\tau\sigma(\omega) = \sigma\tau(\omega) = \sigma(\omega^2) = \tau(\omega)$

$\sigma\tau\sigma(\sqrt[3]{2}) = \sigma\tau(\omega\sqrt[3]{2}) = \sigma(\omega^2\sqrt[3]{2}) = \sqrt[3]{2} = \tau(\sqrt[3]{2})$

hence $\sigma\tau\sigma = \tau \Rightarrow \sigma\tau = \tau\sigma^2$,

therefore $\mathrm{Gal}(K/\mathbb{Q}(\sqrt{3})) \cong \langle \tau, \sigma : \tau^2 = \sigma^3 = 1, \sigma\tau = \tau\sigma^2 \rangle \cong S_3$.

and

now $\mathrm{Gal}(K/\mathbb{Q}(\sqrt[3]{2}, i)) \cong \mathbb{Z}_2$, and

$\mathrm{Gal}(K/\mathbb{Q}(\sqrt[3]{2},\omega)) \cap \mathrm{Gal}(K/\mathbb{Q}(\sqrt{3})) = \mathbb{Z}_2 \cap S_3 = \{1\}$,

fix different subfields, so $\cap = \emptyset$

& $|\mathbb{Z}_2||S_3| = |G|$, i.e. $\boxed{\mathbb{Z}_2 \oplus S_3 \cong G}$

(5) (1) Similary classes of nilpotent elts in $M_4(F)$

For any nilpotent matrix $A \in M_4(F)$, $A^4 = 0$.

Since $Av = \lambda v \Rightarrow 0 = A^4 v = \lambda^4 v \Rightarrow \lambda^4 = 0 \Rightarrow$ eigenvalues all 0.

So eigenvalues are all in F, so apply Jordan Canonical Form:

$\exists S \in GL_4(F)$ such that $SNS^{-1} = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{bmatrix}$ for $a, b, c \in \{0, 1\}$.

$N^2 = 0 \Rightarrow b = 0, a = c = 1$ or only one a, b, c is nonzero $c = 1$

$N^3 = 0 \Rightarrow a = b = 1, c = 0$ or $a = 0, b = c = 1$ $b = c = 0, a = 1$

$N^4 = 0 \Rightarrow a = b = c = 1$.

see that $\begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & ab & 0 \\ 0 & 0 & 0 & bc \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} & " & \end{bmatrix}^3 = \begin{bmatrix} 0 & 0 & 0 & abc \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} & " & \end{bmatrix}^4 = 0$

* All nilp. elts have same char poly

Classes given by:

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{nil deg } 2}, \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{nil deg } 3}, \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\text{nil deg } 4}$$

same min poly, but not conjugates

⑥ $R = \mathbb{Q}[x,y,z]$

(i) Every simple R-module M_R has $\dim_{\mathbb{Q}} M_R < \infty$.

~~Let M be simple R-module.~~

~~Consider M as a $\mathbb{Q}$-module and then~~

~~$M \otimes_{\mathbb{Q}} \mathbb{Q}[x,y,z]$ simple~~ simple.

~~But if M had $\dim_{\mathbb{Q}} M = \infty$, then $M \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \cdots$~~,

~~hence $(\mathbb{Q} \oplus \cdots) \otimes_{\mathbb{Q}} \mathbb{Q}[x,y,z] \cong \bigoplus_i (\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}[x,y,z]) \cong \bigoplus_i \mathbb{Q}[x,y,z]$~~

~~hence an infinite rank R-module, which isn't simple, hence contradiction.~~

A simple R-module M has $M \cong R/I$, I a maximal ideal, hence $M \cong R/I \cong F$ field where $F/\mathbb{Q}$.

Now, R/I is finitely-generated algebraic extension of $\mathbb{Q}$, hence finite $\mathbb{Q}$-dimension.

? By construction

Zariski Theorem

So M has finite $\mathbb{Q}$-dimension

R fin-gen, as since $\mathbb{Q}[x,y,z]$, hence

$\mathbb{Q}[x,y,z] \to R/I$ is surj,

so R/I also fin-gen

⑦ Homogeneous system of m linear equations in n variables over $\mathbb{Z}$ and assume $d = n - m \geq 0$.

(i) show the set of sol'ns is free abelian grp of rank at least d

we work in $\mathbb{Z}[x_1, \dots, x_n]$

$$A\vec{x} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ & \ddots & & \vdots \\ a_{1m} & a_{2m} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{21}x_2 + \dots + a_{n1}x_n \\ \vdots \\ a_{1m}x_1 + a_{2m}x_2 + \dots + a_{nm}x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

So the vectors $\vec{x} \in \mathbb{Z}^n$ in the free module $\mathbb{Z}^n$ such that $Ax = 0$ are the solution sets, ~~hence for $A : \mathbb{Z}^n \to \mathbb{Z}^m$~~ and soln set $= \ker A$.

~~see that $0 \to \ker A \hookrightarrow \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^m \to 0$ (exact if A surj)~~

~~Now $\mathbb{Z}^m$ is free, so projective, so the seq. splits and we get $\mathbb{Z}^n = \mathbb{Z}^m \oplus \ker A$, hence $\ker A$ projective, i.e. the sol'n set is projective, but $\ker A$ a $\mathbb{Z}$-module, hence projective $\Rightarrow$ free.~~

- $\ker A \leq \mathbb{Z}^n$ a subgroup, hence torsion free, hence also free.
- $A : \mathbb{Z}^n \to \mathbb{Z}^m \leq \mathbb{Z}^n$

  hence $\operatorname{rk} \ker A \geq n - m = d$

  since at least $n-m$ summands must go to zero.

(ii) rk of group of sol'ns?

$A : \mathbb{Z}^n \to \mathbb{Z}^m$; let $\operatorname{rk} A = k$

$\to A : \mathbb{Z}^n \to A(\mathbb{Z}^n) = \mathbb{Z}^k \leq \mathbb{Z}^m \leq \mathbb{Z}^n$

and $\mathbb{Z}^n / \ker A \cong A(\mathbb{Z}^n) \cong \mathbb{Z}^{\operatorname{rk} A}$

$n - \operatorname{rk}(\ker A) = \operatorname{rk} A \Rightarrow \underline{n - \operatorname{rk} A = \operatorname{rk} \ker A}$