ALGEBRA QUALIFYING EXAM (MATH 510AB)

FALL 2000

(1) Describe all groups of order $3 \cdot 17 \cdot 23$ up to isomorphism.

(2) Let $G$ be a finitely generated Abelian group so that every proper homomorphic image of $G$ is cyclic. Prove that $G$ is cyclic or that $|G| = p^2$ for $p$ a prime.

(3) Let $K \subseteq \mathbb{C}$ be a splitting field over $\mathbb{Q}$ of $x^5 - 5$. Describe $\mathrm{Gal}(K/\mathbb{Q})$. Describe those fields $\mathbb{Q} \subseteq M \subseteq K$ with $M$ Galois over $\mathbb{Q}$, and for these find $\mathrm{Gal}(M/\mathbb{Q})$.

(4) Let $\overline{F}$ be an algebraic closure of the field $F$. If $M \subseteq F[x_1, \ldots, x_n]$ is a maximal ideal, show that $V(M) = \{a \in \overline{F} \times \cdots \times \overline{F} | f(a) = 0 \text{ for all } f \in M\}$ in finite and not empty.

(5) Let $M \subseteq \mathbb{Q}$ be Noetherian $\mathbb{Z}$-submodule. For $N$ a $\mathbb{Z}$-submodule of $M$, show $M/N$ is finite (as a set) $\Leftrightarrow M \otimes_{\mathbb{Z}} \mathbb{Q} \cong N \otimes_{\mathbb{Z}} \mathbb{Q}$.

(6) If $R$ is a right Artinian ring and $x^3 = x$ for all $x \in R$, show: $R$ is commutative; $R$ is finite; and $R$ has $2^a 3^b$ elements for some $a, b \geq 0$.

$a^2 = 1 \Rightarrow a^{-1} = a$

and $aba^{-1}b^{-1} = abab = (ab)^2 = 1$

$(ab)^2 = 1$
$\|$
$abab$

③ G fin-gen abelian group s.t. every proper homomorphic image of G is cyclic, prove that G is cyclic or $|G| = p^2$

$f : G \to G$ s.t. $G/\ker f \cong f(G) \cong \mathbb{Z}_n$ or $\mathbb{Z}$. (*)

Recall that every normal subgroup has a homomorphism with it as kernel, hence (*) means $G/N \cong \mathbb{Z}_n$ or $\mathbb{Z}$ for every proper normal subgroup $N \trianglelefteq G$.

But G is abelian, hence every s.g. is normal, hence $G/N \cong \mathbb{Z}_n$ or $\mathbb{Z}$ for every s.g. $N \leq G$. ⟶

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~~Suppose $G/N \cong \mathbb{Z}_n$~~ :

~~See that $G \cong A_{p_1} \oplus \cdots \oplus A_{p_k} \oplus \mathbb{Z}^N$, but $G/H \cong \mathbb{Z}_n$ for every subgroup H, hence G must be torsion since if $N \geq 1$, there exists subgroups of G with no free part, hence for such s.g. $G/H$ has free part, contradiction~~

~~Suppose $G/N \cong \mathbb{Z}$~~

~~$G \cong A_{p_1} \oplus \cdots \oplus A_{p_k} \oplus \mathbb{Z}^N$ and $G/N \cong \mathbb{Z}$ or $\mathbb{Z}_n$~~

~~→ So every subgroup must have free part $\geq N-1$;~~

~~If G free, then $N = 1 \Rightarrow$ G cyclic ; $N = 2 \Rightarrow$~~
~~$N > 2 \Rightarrow G/\mathbb{Z} \cong \mathbb{Z}^{N-1}$ ⟶ hence contradiction~~

~~If G torsion, then if $N \geq 1$ there exist torsion subgroups (free part 0), hence contradiction. If $N = 1$, then $G \cong T \oplus \mathbb{Z}$, hence $G/T \cong \mathbb{Z}$ and $G/\mathbb{Z} \cong T$ must be cyclic ; if T & $\mathbb{Z}$ are cyclic, so is G~~

~~finite order (T) free ($\mathbb{Z}$)~~

②cont'd

$G/N$ cyclic for all nontrivial s.g. ∃ $N$ ; WANT $G$ cyclic or $G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$

$G \cong T \oplus \mathbb{Z}^k$

**If G free**: $T = 0$, hence $G/N \cong \mathbb{Z}$ for all subgroups $N$. Hence $N$ must have $rk(N) \geq k-1$, i.e. can have free rk $k-1$ or $k$.

If $k > 2$, ∃ s.g. of rk 1, which is neither $k-1$ or $k$. So $k = 1$ or $2$. If $k=1$, $G$ is cyclic

Suppose $k = 2$. Then $\mathbb{Z} \cong G/H \cong \mathbb{Z} \oplus \mathbb{Z}/H$ hence $H$ must have free rank 1

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If $T \neq 0, k \neq 0$:

If $k > 1$, there exist torsion subgroup $H$, hence $G/H$ has free rank $k > 1$, hence not cyclic, hence contradiction

If $k = 1$ we have $G \cong T \oplus \mathbb{Z}$ and therefore $G/\mathbb{Z} \cong T$ must be cyclic. So $T$ is torsion cyclic & $\mathbb{Z}$ infinite cyclic
→ $T \oplus \mathbb{Z} \cong G$ is cyclic

If $G$ is torsion ($k=0$) suppose $n > 1$.

Then $G \cong A_{p_1} \oplus \cdots \oplus A_{p_n}$ ; therefore $G/H$ will not be infinite cyclic. Consider the subgroup $H_i = A_{p_1} \oplus \cdots \oplus \widehat{A_{p_i}} \oplus \cdots \oplus A_{p_n}$

Hence $A_{p_i} \cong G/H_i$ is cyclic ; hence $G$ is direct sum of cyclic, coprime each → cyclic

Suppose $n=1$, hence $G$ is $p$-primary.

~~Then $G/H \cong \mathbb{Z}_{p^l}$ for some $l$~~

Then $|G| \cong p^l$. ~~and $H \leq G$ with $|H| \cong p^r$ and $G/H \cong \mathbb{Z}_{p^{l-r}}$~~

Suppose $G$ not cyclic. Then $G \cong A \oplus B$, both $p$-primary.

$G/B \cong A$ is cyclic

$G/A \cong B$ is cyclic, hence $G$ is direct sum of two $p$-primary cyclic groups; i.e.

$$G \cong \mathbb{Z}_{p^{l_1}} \oplus \mathbb{Z}_{p^{l_2}}$$

If neither are 1,
then we have $\mathbb{Z}_p \oplus \mathbb{Z}_p \leq G$ and $G/\mathbb{Z}_p \oplus \mathbb{Z}_p \cong \mathbb{Z}_{p^{l_1-1}} \oplus \mathbb{Z}_{p^{l_2-1}}$ not cyclic,

hence one is $=1$.

WLOG, $l_1 = 1$.

$G \cong \mathbb{Z}_p \oplus \mathbb{Z}_{p^{l_2}}$ has sg. $1 \oplus \mathbb{Z}_p \leq \mathbb{Z}_p \oplus \mathbb{Z}_{p^{l_2}} = G$

and $G/1 \oplus \mathbb{Z}_p \cong \mathbb{Z}_p \oplus \mathbb{Z}_{p^{l_2-1}}$ not cyclic. unless $l_2 = 1$.

Hence $l_2 = 1$ and $\underline{G \cong \mathbb{Z}_p \oplus \mathbb{Z}_p}$.

④ $\bar{k}$ alg closure of $k$, $M \subseteq k[x_1, \ldots, x_n]$ maximal

Show $Var_{\bar{k}}(M)$ is finite & nonempty

$M$ maximal $\Rightarrow$ $k[x_1, \ldots, x_n]/M$ a field containing $k$.

This field is finitely-generated as a $k$-algebra, & algebraic over $k$, hence a finite extension, hence a fin-dim'l $k$-vector space.

Now consider the injection:

$$k[x_i]/I \cap k[x_i] \hookrightarrow k[x_1, \ldots, x_n]/I$$

If $I \cap k[x_i] = 0$, then $k[x_i]$ injects into $k[x_1, \ldots, x_n]/I$, contradiction since $k[x_i]$ is infinite dimensional $k$-vector space.

Therefore $I \cap k[x_i] \neq 0$.

But any polynomial in $k[x_i]$ has a finite number of zeros in $\bar{k}$, hence the set of common zeros of $I$ in $\bar{k}$ must be finite.

(5) $M \subseteq \mathbb{Q}$ a noeth. $\mathbb{Z}$-module.

$N$ a $\mathbb{Z}$-submodule of $M$, show $M/N$ finite $\Leftrightarrow M \otimes_{\mathbb{Z}} \mathbb{Q} \cong N \otimes_{\mathbb{Z}} \mathbb{Q}$

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$M$ noetherian $\mathbb{Z}$-module. $\Rightarrow$ Every sub-$\mathbb{Z}$-module is finitely-generated

$\Rightarrow N \cong \mathbb{Z}^k \oplus \mathbb{Z}/(a_1) \oplus \cdots \oplus \mathbb{Z}/(a_m)$

($\Rightarrow$)

$M/N$ finite $\Rightarrow$ $M$ and $N$ have same free rank

$\Rightarrow$ $M$ has finite free rank.

$\Rightarrow M \cong \mathbb{Z}^k \oplus T$

Then $M \otimes_{\mathbb{Z}} \mathbb{Q} \cong (\mathbb{Z}^k \oplus T) \otimes_{\mathbb{Z}} \mathbb{Q} \cong (\mathbb{Z}^k \otimes_{\mathbb{Z}} \mathbb{Q}) \oplus (T \otimes_{\mathbb{Z}} \mathbb{Q})$ [$\to 0$]

$\cong (\mathbb{Z} \otimes \mathbb{Q})^k \cong \mathbb{Q}^k$

$N \otimes \mathbb{Q} \cong (\mathbb{Z}^k \oplus S) \otimes_{\mathbb{Z}} \mathbb{Q} \cong (\mathbb{Z}^k \otimes \mathbb{Q}) \oplus (S \otimes \mathbb{Q})$ [$\to 0$]

($S$: torsion) $\cong \mathbb{Q}^k$

hence $M \otimes \mathbb{Q} \cong N \otimes \mathbb{Q}$

($\Leftarrow$)

If $N \otimes \mathbb{Q} \cong M \otimes \mathbb{Q}$ we have that $N$ and $M$ have the same free rank (which is finite since $N$ is fin-gen).

Then $M \cong \mathbb{Z}^k \oplus T \leftarrow$ maybe not fin-gen
$N \cong \mathbb{Z}^k \oplus S \leftarrow$ fin-gen
$\} \to S \subseteq T$

~~$M/N \cong T/S$~~ noetherian

$M$ noetherian, so every submodule is fin-gen, hence $T$ is finitely generated. and torsion, hence finite.

So $M/N \cong T/S$ is finite

(6) R artinian ring, $x^3 = x$ $\forall x \in R$.
Show: R comm, R finite, and R has $2^9 3^6$ elts

R artinean $\Rightarrow$ $J(R)$ nilpotent
$x^3 = x$ $\forall x \in R \Rightarrow$ R has no nontrivial nilpotent elts } $\Rightarrow J(R) = 0$

so R is JSS & Art $\Rightarrow$ R s.s.

Art-wedd $\Rightarrow R \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_k}(D_k)$

we showed R has no nilp. elts, hence $n_i = 1$ for all $i$.

$\Rightarrow R \cong D_1 \oplus \cdots \oplus D_k$

Now we know $x^3 = x$ in each $D_i$, but division ring, so then $x^{-1}x^3 = x^{-1}x \Rightarrow x^2 = 1 \Rightarrow$ every element is order 2

Now consider $a, b \in D_i$ and see that $a^2 = 1 \Rightarrow a^{-1} = a$

Therefore $aba^{-1}b^{-1} = abab = (ab)^2 = 1$ (since $ab \in D_i$)

Hence $D_i$ is commutative $\forall i$ $\Rightarrow$ R comm.

But comm. division rings are fields, hence the $D_i$ are fields. Now, any finite subgroup of $D_i^\times$ must be cyclic, but every elt has order 2, hence any finite sg. is $\mathbb{Z}_2$ or trivial.

If $D_i^\times$ were infinite, it would be infinite sum of $\mathbb{Z}_2$'s, but can't happen since we get zero divisors. So $D_i^\times \cong \mathbb{Z}_2$ or $0$,

hence $D_i \cong \mathbb{F}_2$ or $\mathbb{F}_3$.

Hence R is direct sum of $\mathbb{F}_2$'s & $\mathbb{F}_3$'s, hence $|R| = 2^9 3^6$