ALGEBRA QUALIFYING EXAM NOVEMBER, 1999

Partial credit is given for partial solutions.

Sylow, solvable gps ~~1.~~ For p and q distinct primes show that any group of order p^2q is solvable.

Fin abelian group 2. Let G be a finite Abelian group so that whenever H and K are subgroups of G of the same order then $H \cong K$ as groups. Describe the possible structures of G. If $|G| = 2^3 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$, up to isomorphism how many possibilities are there for G?

Galois theory 3. Let $p_1, \ldots, p_k$ be distinct primes in $\mathbf{Z}$ and set $F = \mathbf{Q}(\sqrt{p_1}, \sqrt{p_2}, \ldots, \sqrt{p_k}) \subseteq \mathbf{R}$.

i) Show that F is a Galois extension of $\mathbf{Q}$ with $\mathrm{Gal}(F/\mathbf{Q}) \cong (\mathbf{Z}/2\mathbf{Z})^k$.

ii) Show that $F = \mathbf{Q}(\sqrt{p_1} + \cdots + \sqrt{p_k})$.

Modules, sequences 4. For F a field and $R = F[x_1, \ldots, x_n]$ let M be a finitely generated R module. Show that there are positive integers s and t and an exact sequence of R modules $0 \to \underset{\ker\alpha}{K} \to R^s \xrightarrow{\alpha} R^t \xrightarrow{\beta} M \to 0$.

Nullstellensatz 5. If I is a nonzero ideal of $R = \mathbf{C}[x_1, \ldots, x_n]$ which is not maximal then if R/I is a domain, show that $\dim_{\mathbf{C}} R/I$ must be infinite. — R/I fin-dim over $\mathbb{C}$ ⇒ field

Artin-Wedderburn. 6. If $R \neq \{0\}$ is a finite ring so that each $r \in R$ satisfies the polynomial $x^8 = x$, describe the possible structures of R.

$I \subseteq R = \mathbb{C}[x_1,\ldots,x_n]$ not maximal. If R/I domain show $\dim_{\mathbb{C}} R/I$ infinite.

$\sqrt{I} = Id(Var(I))$

I not maximal, not of form $(x_1-a_1,\ldots,x_n-a_n)$

(R/I domain ⇒ I prime ⇒ $\sqrt{I} = I$

$I = Id(Var(I))$

Suppose R/I findim'l/$\mathbb{C}$

But a domain over a field and algebraic findim'l

hence field, hence I max ↯.

Algebra
Nov '99

① p, q primes; Show $|G| = p^2 q$ solvable.

Case $p > q$: $r_p \equiv 1 \pmod{p}$ & $r_p | q \Rightarrow r_p = 1, \cancel{q}$

hence the Sylow p-subgroup is normal; call it P.

Recall that a p-group is solvable, hence P solvable, but $P \trianglelefteq G$, hence $G/P \cong Q$, q-subgroup, which is also a p-group, hence solvable.

P & G/P solvable $\Rightarrow$ G solvable.

Case $p < q$: $r_p \equiv 1 \pmod{p}$ & $r_p | q \Rightarrow r_p = 1, q$.

$r_q \equiv 1 \pmod{q}$ & $r_q | p^2 \Rightarrow r_q = 1, \cancel{p}, p^2$

If $r_q = 1$, then precede as before.

Suppose $r_q > 1$. Then $r_q = 1 + kq$ and $r_q = p^2$

$\Rightarrow kq = p^2 - 1 = (p-1)(p+1)$

$\Rightarrow q | (p-1)(p+1) \Rightarrow q | (p-1)$ or $q | (p+1)$ since q prime.

$q > p$, hence $q \nmid (p-1)$, hence $q | (p+1)$.

But again since $q > p$, we have $q = p + 1$, hence $\begin{cases} q = 3 \\ p = 2 \end{cases}$,

hence $|G| = 12$, since we suppose $r_3 = 4$; we have $4 \cdot 2 = 8$ elements order 3, leaving only 4 other elts, hence the Sylow 2-s.g. is normal, and proceed as before.

④ F field, $R = F[x_1, \dots, x_n]$, M fin-gen R-mod.

Show $\exists$ $s,t > 0$ s.t. $0 \to R \to R^s \xrightarrow{\alpha} R^t \xrightarrow{\beta} M \to 0$ exact

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Let S be set of generators of M. Then let $R[S]$ be the free module on the generators.

and consider the map $\varphi : R[S] \to M$
$$s \in S \mapsto s$$

Then

$$0 \to \ker\varphi \to R[S] \to M \to 0 \text{ is exact.}$$

$$0 \to (\ker\alpha) \hookrightarrow R^s \xrightarrow{\alpha} R^t \xrightarrow{\beta} M \to 0$$

surjective

$\operatorname{im}\alpha = \ker\beta$

(5) $I \subseteq R = \mathbb{C}[x_1, \dots, x_n]$ not maximal

Then if R/I domain, show $\dim_{\mathbb{C}} R/I = \infty$

R/I domain $\Rightarrow I$ prime $\Rightarrow I = \sqrt{I} = Id(Var(I))$.

$= \bigcap_{\alpha} (x_1 - \alpha_1, \dots, x_n - \alpha_n)$

intersection of maximal ideals.

know $\mathbb{C}[x_1, \dots, x_n]/\sqrt{I} \cong \mathbb{C}^{|Var\, I|}$

$\cong$

$\mathbb{C}[x_1, \dots, x_n]/I$

~~$\mathbb{C}[x_1, \dots, x_n]$ noetherian $\Rightarrow I = \langle f_1, \dots, f_t \rangle$~~

Suppose finite-dimensional; then $|Var\, I|$ is finite, hence $I = \bigcap_{i=1}^{k} (x_1 - \alpha_1, \dots, x_n - \alpha_n)$ finite intersection of max. ideals.