Written Qualifying Exam, Algebra, Nov. 1998

Directions. Partial credit in units of 1/4 is given for partial solutions.

Solvable gps ~~1.~~ Let G be a group of order $p^a q^b$, p, q distinct primes and a, b positive integers. Prove that if $q < p$ and the order of q mod p exceeds b then G is solvable.

2. Let G be a finitely generated abelian group (i.e., a finitely generated $\mathbb{Z}$-module).

Modules over PID ~~a).~~ Prove that G has no elements of order p, p a prime, if and only if $G \otimes_{\mathbb{Z}} \mathbb{Z}_P \cong \mathbb{Z}_P^r$ for some positive integer r, $\mathbb{Z}_P$ is the local ring of rational numbers with denominator prime to p.

(Proj. modules) ~~b).~~ Prove that G is projective if and only if there is an integer r such that $G \otimes_{\mathbb{Z}} H \cong H^r$ for all abelian groups H.

Galois NORM & TRACE

3. Let $\mathbb{F}_{p^n}$ be a finite field with p^n elements, p a prime. Recall that the norm map $N : \mathbb{F}_{p^n}$ is defined by $N(x) = \prod_{g \in Gal_{\mathbb{F}_p}\mathbb{F}_{p^n}} g(x)$ and the trace map is defined by $T(x) = \sum_{g \in Gal_{\mathbb{F}_p}\mathbb{F}_{p^n}} g(x)$. Determine the image of each of these maps, show that the kernel of the norm map is $\{x/g(x) : x \in \mathbb{F}_{p^n}^{\times}, g \in Gal_{\mathbb{F}_p}\mathbb{F}_{p^n}\}$ and that the kernel of the trace map is $\{x - g(x) : x \in \mathbb{F}_{p^n}, g \in Gal_{\mathbb{F}_p}\mathbb{F}_{p^n}\}$.

Nullstellensatz 4. Let R be a subring of $\mathbb{C}[x_1, \ldots, x_n]$ containing $\mathbb{C}$ and assume that the field of quotients of R is $\mathbb{C}(x_1, \ldots, x_n)$. Show that there are polynomials $f_1, \ldots, f_x \in \mathbb{C}[x_1, \ldots, x_n]$ such that $d\mathbb{C}[x_1, \ldots, x_n] \subset R$ if and only if $d \in I = f_1\mathbb{C}[x_1, \ldots, x_n] + \cdots + f_s\mathbb{C}[x_1, \ldots, x_n]$. In addition, show that I cannot be a maximal ideal in $\mathbb{C}[x_1, \ldots, x_n]$.

(Artin-Wedd) 5. Maschke's theorem implies that the group algebra $k[G]$ over a field k of characteristic zero is semisimple when G is a finite group. Using this fact,

a). Determine the structure of $\mathbb{C}[S_3]$, S_3 the symmetric group on three symbols.

b). An epimorphism of groups, $\phi : G \to H$, induces an epimorphism $\Phi : k[G] \to k[H]$ on the corresponding group rings over k. Prove that if k has characteristic 0 and G is finite then $k[H]$ is a ring direct summand of $k[G]$.

Galois group 6. Determine the galois group of $x^4 - p$ over the rationals, p a prime, and determine all subfields of its splitting field which are normal over the rational numbers.

$m \otimes n \in \mathbb{Z}_p \otimes \mathbb{Z}_q$

$p(m \otimes n) = pm \otimes n = 0$

$= m \otimes pn \Rightarrow m = 0$ or $pn = 0$

$\Rightarrow m = 0$ or $n = 0$

$\Rightarrow m \otimes n = 0$

Fall '98 Algebra:

① $|G| = p^a q^b$, p, q primes, $a, b > 0$, $q < p$, $q^k \not\equiv 1 \pmod{p}$ for all $k \leq b$

~~Show solvable:~~

$r_p \equiv 1 \pmod{p}$ & $r_p \mid q^b \Rightarrow r_p \equiv 1, \cancel{q, q^2, \ldots, q^b}$

$\Rightarrow r_p \equiv 1$

$\Rightarrow$ the Sylow p-sg. is normal, call it $\underline{P}$.

Hence then

$G/\underline{P}$ is a q-group, hence solvable since all p-grps solvable.

Likewise, $\underline{P}$ a p-grp, hence solvable.

$\therefore G/P$ & P solvable $\Rightarrow G$ solvable

② G fin-gen $\mathbb{Z}$-module,

(a) Prove G has no elts order p $\iff$ $G \otimes \mathbb{Z}_p \cong \mathbb{Z}_p^r$ for some $r>0$. Here $\mathbb{Z}_p$ = local ring of rational #'s denom. prime to p

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$G \cong \mathbb{Z}^k \oplus A_p \oplus A_t$. WTS $\underline{A_p = 0}$.

$\cancel{\mathbb{Z}_p \cong \mathbb{Q}}$

$(\Leftarrow)$: $\mathbb{Z}_p^r \cong (\mathbb{Z}^k \oplus A_p \oplus A_t) \otimes \mathbb{Z}_p$

$\cong (\mathbb{Z}^k \otimes \mathbb{Z}_p) \oplus (A_p \otimes \mathbb{Z}_p) \oplus (A_t \otimes \mathbb{Z}_p)$ order is prime to p.

$\cong \mathbb{Z}_p^k \oplus (A_p \otimes \mathbb{Z}_p) \oplus 0$

$|A_t| = m$, $\gcd(m,p) = 1$

$t \otimes \frac{p}{q} \in A_t \otimes \mathbb{Z}_p \Rightarrow t \otimes \frac{mp}{mq}$ ← OK ✓

$\Rightarrow mt \otimes \frac{p}{mq} = 0 \otimes \frac{p}{mq} = 0$ ✓

$\cong \mathbb{Z}_p^k \oplus (A_p \otimes \mathbb{Z}_p)$

↑ nontrivial torsion $\mathbb{Z}$-module. unless $A_p = 0$.

$\mathbb{Z}/(p) \otimes \mathbb{Z}_p$

hence $A_p = 0$, and thus no elts order p.

$(\Rightarrow)$ similar.

(3) $\overset{E/k}{\mathbb{F}_{p^n}/\mathbb{F}_p}$, $G = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) = \langle \sigma : x \mapsto x^p \rangle$

Recall $N(x) = \prod_{g\in G} g(x)$ & $T(x) = \sum_{g\in G} g(x)$.

Determine the images and show that

$\ker N = \{x/g(x) : x \in \mathbb{F}_{p^n}^\times, g \in G\}$

$\ker T = \{x - g(x) : x \in \mathbb{F}_{p^n}, g \in G\}$

So $N(x) = \prod_{k=1}^{n} \sigma^k(x)$, $T(x) = \sum_{k=1}^{n} \sigma^k(x)$

Both have codomain k

T surjective by linearity and linear independence of characters.

$\alpha \in E^\times$.

$N(\alpha) = \prod_{g\in \mathrm{Gal}} g(\alpha)$

$h \in \mathrm{Gal}$; $h(N(\alpha)) = h(\prod_{g\in\mathrm{Gal}} g(\alpha))$ left mult is surjective from grp to grp

$= \prod_{g\in\mathrm{Gal}} hg(\alpha)$

$= \prod_{g\in\mathrm{Gal}} g(\alpha)$

$= N(\alpha)$, h arbitrary, so $N(\alpha)$ fixed by all members of Gal grp

$\Rightarrow$ $\underline{N(\alpha) \in k^\times}$

(b) G proj. $\Leftrightarrow$ $\exists r$ s.t. $G \otimes_{\mathbb{Z}} H \cong H^r$ for all abelian grps H

G module over PID, hence projective $\Leftrightarrow$ free.

($\Rightarrow$)

Suppose G proj; then $G \cong \mathbb{Z}^k$.

Let H be abelian; then $G \otimes_{\mathbb{Z}} H \cong \mathbb{Z}^k \otimes_{\mathbb{Z}} H$

hence $\mathbb{Z}$-mod

$$\cong (\mathbb{Z} \otimes_{\mathbb{Z}} H)^k$$

$$\cong H^k.$$

($\Leftarrow$) G fin-gen abelian, hence $G \cong \mathbb{Z}^k \oplus G_t$.

Suppose $G \otimes_{\mathbb{Z}} H \cong H^r$ for all $\mathbb{Z}$-mod H.

$$\rightsquigarrow (\mathbb{Z}^k \oplus G_t) \otimes H \cong (\mathbb{Z}^k \otimes_{\mathbb{Z}} H) \oplus (G_t \otimes_{\mathbb{Z}} H)$$

$$\cong H^k \oplus (G_t \otimes_{\mathbb{Z}} H)$$

hence $G_t \otimes H = 0$ for all H, hence $G_t = 0$,

hence G free, hence proj

④ $R \subseteq \mathbb{C}[x_1,\dots,x_n]$ subring w/ $\mathbb{C}$ and $\mathrm{Frac}(R) = \mathbb{C}(x_1,\dots,x_n)$.

Show that $\exists f_1,\dots,f_s \in \mathbb{C}[x_1,\dots,x_n]$ s.t.

$$d\,\mathbb{C}[x_1,\dots,x_n] \subseteq R \iff d \in I = f_1\mathbb{C}[x_1,\dots,x_n] + \dots + f_s\mathbb{C}[x_1,\dots,x_n].$$

($\Rightarrow$) Suppose we have $d \in \mathbb{C}[x_1,\dots,x_n]$ s.t. $(d) \subseteq R$.

<u>WTS: R noetherian.</u>

$\hookrightarrow (d) = (f_1,\dots,f_s) = (f_1) + \dots + (f_s)$

$\rightarrow$ $\underline{d \in (f_1) + \dots + (f_s)}$.

suppose x s.t. $N(x) = 0$.

$$\prod_{k=1}^{n} \sigma^k(x) = \prod_{k=1}^{n} x^{p^k} = x^p x^{p^2} \cdots x^{p^n}$$

$$= x^{p+p^2+\cdots+p^n}$$

$$= x^{1+p+p^2+\cdots+p^{n-1}} \qquad = x^{1-p^k}$$

$\cancel{0 = x^{1+p+\cdots+p^{n-1}}}$

$\cancel{= x^{\frac{p^n-1}{p-1}}}$

$p^n - 1 = \cancel{(p-1)(p^{n-1}+p^{n-2}+\cdots+p+1)}$

$\cancel{= \frac{x^{\frac{p^n}{p-1}}}{x^{\frac{1}{p-1}}}}$

$$= \left(\frac{x}{x^{p^k}}\right)^{1+\cdots+p^{n-1}}$$

$$= \frac{x^{1+\cdots+p^{n-1}}}{x^{p^k(1+\cdots+p^{n-1})}}$$

$$= 1$$

$\cancel{p^3(1+p+p^2+p^3+p^4+p^5)}$

$\cancel{p^3+p^4+p^5+p^6+p+p^2}$

$\cancel{p^2(1+p+p^2+p^3)}$

$= \cancel{p^2+p^3+1+p}$