ALGEBRA QUALIFYING EXAM **MAY, 1997**

$99 = 3^2 \cdot 11 \cdot$

Partial credit is given for partial solutions.

$5\overline{)495}$
450
45
45
0

Group Classify ~~1.~~ Up to isomorphism, describe all group of order 495. $= 3^2 \cdot 5 \cdot 11$

Galois Group ~~2.~~ Let $x^4 - 7 \in F[x]$ for $F \subseteq C$. If $F \subseteq M \subseteq C$ and M is a splitting field for $x^4 - 7$ over F, find Gal(M/F): when F = Q; when $F = Q[\sqrt{7}]$; and when F = Q[i], with $i^2 = -1$.

Module over PID ~~3.~~ Let M be a finitely generated F[x] module (F a field). If every submodule of M has a complement, describe the structure of M in terms of F[x]. (Recall that a submodule H of a module M has a complement if there is a submodule H' so that $M \cong H \oplus H'$; i.e. $H + H' = M$ and $H \cap H' = (0)$.)

Nullstell. ~~4.~~ Show that some power of $(x + y)(x^2 + y^4 - 2)$ is in the ideal of C[x,y] generated by $x^3 + y^2$ and $y^3 + yx$.

5. Let R be a commutative Noetherian ring with no nonzero nilpotent element. Set A = {ann I | I is a nonzero ideal of R} and M = {maximal elements in A}. Prove that R embeds in a direct sum of finitely many domains as follows:

a) Show that the elements of M are prime ideals in R.

b) For $P \neq Q$ in M, show ann $Q \subseteq P$.

c) Show that M is finite (consider sums of ann P_i for $P_i \in M$).

d) Show that the intersection of the elements in M is zero.

Artin Wedderburn 6. Let R be a finite dimensional algebra over the field F. Assume that for every $r \in R$ there some $g(x) \in F[x]$, depending on r, so that $r + g(r)r^2 = 0$. Determine the structure of R.

$5 \cdot 11 \cdot 8$

$(3^2 \cdot 5 \cdot 11) - (8 \cdot 5 \cdot 11)$
$= 5 \cdot 11(9-8) = 55$

Algebra: Spr 97

① $|G| = 495 = 3^2 \cdot 5 \cdot 11$

Sylow $r_{11} \equiv 1 \pmod{11}$ & $r_{11} \mid 3^2 \cdot 5 \Rightarrow r_{11} = 1, \cancel{3}, \cancel{3^2}, \cancel{3 \cdot 5}, 3^2 \cdot 5, \cancel{5}$

$r_5 \equiv 1 \pmod 5$ & $r_5 \mid 3^2 \cdot 11 \Rightarrow r_5 = 1, \cancel{3}, \cancel{3^2}, \cancel{3 \cdot 11}, \cancel{3^2 \cdot 11}, 11$

$r_3 \equiv 1 \pmod 3$ & $r_3 \mid 5 \cdot 11 \Rightarrow r_3 = 1, \cancel{5}, \cancel{11}, 5 \cdot 11$

- Suppose $r_{11} = 3^2 \cdot 5$: Then there are $10 \cdot 3^2 \cdot 5$ elts order 11, hence $(3^2 \cdot 5 \cdot 11 - 3^2 \cdot 5 \cdot 10) = 3^2 \cdot 5 = 3^2 \cdot 5 = 45$ elts other orders.

  Suppose $r_3 = 5 \cdot 11$; then there are $8 \cdot 5 \cdot 11 = 8 \cdot 55$ elts of order power of 3, but $8 \cdot 55 > 45$, hence $r_3 = 1$, P the Sylow 3-sg normal.

  N an 11 sg. and $P \trianglelefteq G$ Syl 3-sg. $\Rightarrow NP \leq G$. order $11 \cdot 3^2$

- Suppose $r_{11} = 1$: Then N is normal, choose P sylow 3-s.g. and then $NP \leq G$ again

So we can always obtain a s.g. of index 5; so apply regn on cosets: $\varphi: G \to S_5$ with $NP \leq \ker\varphi$.

So $|G/\ker\varphi| \,\big|\, 5!$ & $|G/\ker\varphi| \,\big|\, |G| = 3^2 \cdot 5 \cdot 11 \Rightarrow |G/\ker\varphi| \,\big|\, 5$

$\Rightarrow |G/\ker\varphi| = 5$ since nontrivial $\Rightarrow |\ker\varphi| = 3^2 \cdot 11 \Rightarrow NP = \ker\varphi$, and NP is normal. So now we have, for Sylow 5-s.g. Q, $NP \cap Q = 0$, hence $NPQ \cong G$, hence $G \cong NP \rtimes Q$ and get hom. $\varphi: Q \longrightarrow Aut(NP)$

See that $|NP| = 3^2 \cdot 11$ & $r_{11} = 1 \pmod{11}$ & $r_{11} \mid 3^2 \Rightarrow r_{11} = 1$

$r_3 = 1 \pmod 3$ & $r_3 \mid 11 \Rightarrow r_3 = 1$

$\Rightarrow NP \cong N \oplus P \cong \mathbb{Z}_{11} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$ or $\mathbb{Z}_{11} \oplus \mathbb{Z}_9$

So then $\varphi : Q \longrightarrow \mathrm{Aut}(NP) \cong \mathrm{Aut}(N) \oplus \mathrm{Aut}(P)$

order 5 $\qquad \cong \mathbb{Z}_{10} \oplus \mathrm{Aut}(P)$

$N \cong \mathbb{Z}_{11}$

$P \cong \mathbb{Z}_9$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_3$

case $P = \mathbb{Z}_9$: $\varphi : \underset{\langle q \rangle}{Q} \to \underset{\langle a \rangle}{\overset{\mathbb{Z}_{10}}{\mathrm{Aut}(N)}} \oplus \underset{\langle b \rangle}{\overset{(\mathbb{Z}_9)^\times \cong \mathbb{Z}_{(3)(3-1)} \cong \mathbb{Z}_6}{\mathrm{Aut}(P)}}$

If $\varphi(q)$ is order 1, then $\underline{G \cong \mathbb{Z}_5 \oplus \mathbb{Z}_{11} \oplus \mathbb{Z}_9}$

If $\varphi(q)$ is order 5, then it must map to $(a^2, 1)$ — only order 5 elt in $\mathrm{Aut}(NP)$

Let $N \oplus P \cong \langle \alpha \rangle \oplus \langle \beta \rangle$ $(\alpha^{11} = 1, \beta^9 = 1)$

Then $\theta(\alpha, \beta) = (\alpha^3, \beta)$

here $\varphi(q)(\alpha, \beta) = (\alpha^3, \beta)$.

So $\underline{G \cong \langle q, \alpha, \beta : q^5 = \alpha^{11} = \beta^9 = 1,\ q\alpha q^{-1} = \alpha^3,\ q\beta q^{-1} = \beta \rangle}$

$k^5 = 1 \pmod{11}$

$3^5 = 27 \cdot 9$

$= 5 \cdot 9 = 45$

$\equiv 1 \pmod{11}$

so $k = 3$.

case $P = \mathbb{Z}_3 \oplus \mathbb{Z}_3$: $\varphi : Q \longrightarrow \underset{\mathbb{Z}_{10} \cong \langle a \rangle}{\mathrm{Aut}(N)} \oplus \underset{GL_2(\mathbb{F}_3)}{\mathrm{Aut}(\mathbb{Z}_3 \oplus \mathbb{Z}_3)}$

order 5 elts in $GL_2(\mathbb{F}_3)$? See that $|GL_2(\mathbb{F}_3)| = (3^2 - 1)(3^2 - 3)$

$= (3-1)(3+1)3(3-1)$

$= 2 \cdot 2^2 \cdot 3 \cdot 2 = 2^4 \cdot 3$

$\Rightarrow$ no elts order 5.

So we get the same order 5 elt

$\theta(\alpha, \beta, \gamma) = \theta(\alpha^3, \beta, \gamma)$.

and $\underline{G \cong \langle q, \alpha, \beta, \gamma \mid q^5 = \alpha^{11} = \beta^3 = \gamma^3 = 1,\ q\alpha q^{-1} = \alpha^3,\ q\beta q^{-1} = \beta,\ q\gamma q^{-1} = \gamma \rangle}$

If order 1: $\underline{G \cong \mathbb{Z}_{11} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3}$

② $x^4-7 \in F[x]$, $F \subseteq \mathbb{C}$, $F \subseteq M \subseteq \mathbb{C}$ w/ M/F spl. field for x^4-7.
Find Gal(M/F) when (a) $F=\mathbb{Q}$, (b) $F=\mathbb{Q}(\sqrt{7})$, (c) $F=\mathbb{Q}(i)$.

$f(x)=x^4-7=(x^2-\sqrt{7})(x^2+\sqrt{7})=(x-\sqrt[4]{7})(x+\sqrt[4]{7})(x-i\sqrt[4]{7})(x+i\sqrt[4]{7})$

So then $\mathbb{Q}(i,\sqrt[4]{7})$ is a splitting field for f over $\mathbb{Q}$.

Consider tower

(a) $F=\mathbb{Q}$

$$\begin{array}{ll} \mathbb{Q}(i,\sqrt[4]{7})=M & \\ \quad | & \big) 4 \\ \mathbb{Q}(i) & \\ \quad | & \big) 2 \\ \mathbb{Q} & \end{array}$$

since x^4-7 min poly over $\mathbb{Q}$
(since $\sqrt{7}\notin\mathbb{Q}$ & none of roots are in $\mathbb{Q}$)

Therefore $[M:\mathbb{Q}]=8$, hence $|\text{Gal}(M/\mathbb{Q})|=8$

Then consider $\sigma: \sqrt[4]{7} \mapsto i\sqrt[4]{7}$, $i \mapsto i$ so $\underline{\sigma^4=id}$ $\quad$ $\tau: \sqrt[4]{7}\mapsto\sqrt[4]{7}$, $i \mapsto i^3=-i$ $\rightarrow$ $\underline{\tau^2=id}$

$\tau\sigma\tau(\sqrt[4]{7})=\tau\sigma(\sqrt[4]{7})=\tau(i\sqrt[4]{7})=-i\sqrt[4]{7}=\sigma^3(\sqrt[4]{7})$
$\tau\sigma\tau(i)=\tau\sigma(-i)=\tau(-i)=i=\sigma^3(i)$

hence $\tau\sigma\tau=\sigma^3$ hence.

$\text{Gal}(M/\mathbb{Q}) \cong \langle \sigma,\tau : \sigma^4=\tau^2=1, \tau\sigma\tau=\sigma^3=\sigma^{-1}\rangle$
$\cong D_8$.

(b) $F=\mathbb{Q}(\sqrt{7})$.

Tower:

$$\begin{array}{ll} \mathbb{Q}(i,\sqrt[4]{7}) & \\ \quad | & \big) 2 \text{ since min poly of } \sqrt[4]{7} \text{ over } \mathbb{Q}(i,\sqrt{7}) \text{ is } x^2-\sqrt{7} \\ \mathbb{Q}(i,\sqrt{7}) & \\ \quad | & \big) 2 \\ \mathbb{Q}(\sqrt{7}) & \end{array}$$

so $|\text{Gal}(M/F)|$ 4.

Now consider:

$$\sigma: \sqrt[4]{7} \mapsto -\sqrt[4]{7} \quad \tau: \sqrt[4]{7} \longrightarrow \sqrt[4]{7}$$
$$i \mapsto i \qquad\qquad i \mapsto -i \quad \Big\} \text{ fix } \mathbb{Q}(\sqrt{7})$$

Then $\sigma^2 = id$, $\tau^2 = id$ and distinct ; only are elt of order 2 in $\mathbb{Z}_4$, hence $\underline{Gal(M/\mathbb{Q}(\sqrt{7})) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2}$.

(c) $\underline{F = \mathbb{Q}(i)}$:

$$\begin{array}{c} M = \mathbb{Q}(i, \sqrt[4]{7}) \\ | \\ \mathbb{Q}(i) \end{array} \Big\} \, 4 \quad \text{by logic in pt (a).}$$

Consider $\sigma: \sqrt[4]{7} \mapsto i\sqrt[4]{7}$; clearly in $Gal(M/\mathbb{Q}(i))$
$i \mapsto i$ and order 4,
hence $\underline{Gal(M/\mathbb{Q}(i)) \cong \mathbb{Z}_4}$

③ M fin. gen $F[x]$-module. & every submodule of M has a complement. Describe structure of M:

$F[x]$ a PID, so may use Fund thm of mod. over PID so that we see $M \cong F[x]^k \oplus T$, free & torsion parts.

Now, $T \cong F[x]/(p_1^{e_1}) \oplus \cdots \oplus F[x]/(p_n^{e_n})$

Suppose $e_i > 1$; then there is submodule $F[x]/(p_i) \subseteq F[x]/(p_i^{e_i})$ that does not have a complement.

Hence $e_i = 1 \ \forall i$, and $T \cong F[x]/(p_1) \oplus \cdots \oplus F[x]/(p_n)$.

So: $M \cong F[x]^k \oplus F[x]/(p_1) \oplus \cdots \oplus F[x]/(p_n)$

④ Show some power of $(x+y)(x^2+y^4-2)$ is in $\underbrace{(x^3+y^2, y^3+yx)}_{= I} \subseteq \mathbb{C}[x,y]$.

We therefore want to show that

$$(x+y)(x^2+y^4-2) \in \sqrt{(x^3+y^2, y^3+yx)} = Id(Var(x^3+y^2, y^3+yx))$$

Now, for $(x,y) \in Var(x^3+y^2, y^3+yx)$, get $\begin{cases} x^3+y^2=0 \\ y^3+yx=0 \end{cases}$

$$\Rightarrow \begin{array}{l} x^3=-y^2 \\ y^3=-yx \end{array} \Rightarrow \begin{array}{l} x^3=-y^2 \\ y^2=-x \end{array} \Rightarrow x^3=-x \Rightarrow x^2=-1$$

$$\Rightarrow x = \pm i$$

$$\Rightarrow \begin{cases} y^2 = i \\ y^2 = -i \end{cases} \Rightarrow \begin{cases} y = \pm\sqrt{i} \Rightarrow \begin{cases} (i, \sqrt{i}) \\ (i, -\sqrt{i}) \end{cases} \\ y = \pm i\sqrt{i} \Rightarrow \begin{cases} (-i, i\sqrt{i}) \\ (-i, -i\sqrt{i}) \end{cases} \end{cases} \in Var(I)$$

See that $i^2+(\sqrt{i})^4 = -1+i^2 = -1+-1 = -2$

$i^2+(-\sqrt{i})^4 = -1+i^2 = -2$

$(-i)^2+(i\sqrt{i})^4 = -1+1\cdot(-1) = -2$

$(-i)^2+(-i\sqrt{i})^4 = -1+-1 = -2$

Also, if $x=0 \Leftrightarrow y=0$, hence $(0,0) \in Var(I)$

but $(0+0)=0$

So $(x+y)(x^2+y^4-2)=0$ for all points in $Var(I)$,

hence $\in Id(Var(I)) \Rightarrow \underline{\in \sqrt{I}}$ ✓

⑤ R commutative noeth ring with no nonzero nilpotent elt.

Let $A = \{\text{ann } I : I \subsetneq R \text{ nonzero ideal}\}$

$\mathcal{M} = \{\text{maximal elts. in } A\}$.

Prove that R embeds into direct sum of finitely many domains

(a) Elements of $\mathcal{M}$ are prime ideals in R:

let $M \in \mathcal{M}$. Then $M = \text{ann } I$ some I.

Suppose $xy \in M = \text{ann } I \Rightarrow xyI = 0 \Rightarrow x \in \text{ann}(yI)$

and note that $\underset{M}{\underset{\shortparallel}{\text{Ann}(I)}} \subseteq \text{ann}(yI)$ (since $xI = 0 \Rightarrow xyI = 0$ since $yI \subseteq I$)

But M was maximal, so $\text{ann}(I) = \text{ann}(yI)$

$\Rightarrow x \in \text{ann}(I) = M$

$\Rightarrow$ M prime ideal.

(b) $P \neq Q$ in M; show $\text{ann } Q \subseteq P$:

$x \in \text{Ann}(Q)$; $P \neq Q \Rightarrow \exists q \in Q \setminus P \Rightarrow xq = 0$

$\Rightarrow xq \in P$, but $q \notin P$,

hence $x \in P$ since P prime.

So $\text{Ann}(Q) \subseteq P$.

(c) Show M is finite:

Recall $\text{ann } P \subseteq \text{ann } P + \text{ann } Q$

Let $\{P_i\} \subseteq M$; then $\text{ann } P_1 \subseteq \text{ann } P_1 + \text{ann } P_2 \subseteq \cdots \subseteq \sum_{i=1}^{n} \text{ann } P_i$

But R noeth, hence $\exists n$ s.t.

$\sum_{i=1}^{n} \text{ann } P_i = \sum_{i=1}^{n+1} \text{ann } P_i = \cdots$ By part (b)

$\Rightarrow \text{ann } P_{n+1} \subseteq \sum_{i=1}^{n} \text{ann}(P_i) \subseteq P_{n+1}$, hence $\text{ann } P_{n+1} \subseteq \bigcap P_i$