ALGEBRA QUALIFYING EXAM MAY, 1996

Partial credit is given for partial solutions.

Sylow thms

~~1.~~ Let G be a finite group and p a prime number. Let P be a p-Sylow subgroup of G and denote the normalizer of P in G by $N_G(P)$.

i) Show that $N_G(P) = N_G(N_G(P))$.

ii) If K is a normal subgroup of G and K contains P, show that $G = KN_G(P)$.

iii) If no proper subgroup of G is its own normalizer, show that the center of G is not trivial.

Fin gen Abelian grps + torsion

~~2.~~ Up to isomorphism describe all finitely generated Abelian groups which satisfy all of the following properties: i) $G \otimes_Z Q \cong Q^2$; ii) $G \otimes_Z (Z/7Z) \cong (Z/7Z)^3$; and iii) for any prime $p \neq 7$, $G \otimes_Z (Z/pZ) \cong (Z/pZ)^2$.

3. Let R be a left Artinian ring with Jacobson radical J(R). If $R \neq J(R)$ show that R is a left Noetherian ring.

irreducible polynomials

4. Determine if each of the following polynomials is irreducible, and justify your answer.

i) $x^{2^n} + 1 \in Q[x]$. — Sub $X \to X+1$ and use Eisenstein

ii) $x_n^{\,n} + x_{n-1}^{\,n-1} + \cdots + x_2^{\,2} + x_1 \in F[x_1,\dots,x_n]$ for F any field.

iii) $x^4 + 1 \in F_p[x]$, p an odd prime (note that $p^2 \equiv 1 \pmod 8$).

iv) $x^p + x^{p-1} + \cdots + x + 1 \in F_p[x]$, p an odd prime.

F_2 $(x^{2^n}+1) = (x+1)^{2^n}$

Projective mods.

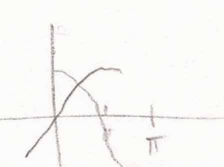
~~5.~~ Let R be a commutative ring with 1, and let $r_1, \dots, r_n \in R$ satisfy $R = Rr_1 + \cdots + Rr_n$. If $M = \{(a_1, \dots, a_n) \in R^n \mid a_1r_1 + \cdots + a_nr_n = 0\}$, show that M is a projective R module.

F_3 $(x^{2^n}+1)$

Galois Theory + Subgroups of A4

~~6.~~ Let K be a finite Galois extension of Q with $Gal(K/Q) \cong A_4$. How many subfields does K contain, what are their dimensions over Q, and which are Galois over Q?

$i^{2^n} = (i^2)^{2^{n-1}} = (-1)^{2^{n-1}} = 1$



$\omega^{2^n} = -1$

$x^{2^{n-1}} = \frac{\pm\sqrt{4}}{2} = \pm 1$

$x^{2^n} = -1$

$\Rightarrow e^{\frac{i\pi}{2^n}} = \cos\left(\frac{\pi}{2^n}\right) + i\sin\left(\frac{\pi}{2^n}\right)$

$x^{2^{n-2}} =$

① $|G|<\infty$, P a Sylow p-s.g., $N_G(P)$ normalizer

(i) Show $N_G(P) = N_G(N_G(P))$:

See that $P \trianglelefteq N_G(P) \trianglelefteq N_G(N_G(P))$. Choose $g \in N_G(N_G(P))$. Since $P \leq N_G(P)$ is a Sylow p-subgroup of $N_G(P)$ as well and $N_G(P)$ is closed under conjugation with g since $N_G(P) \trianglelefteq N_G(N_G(P))$, gPg^{-1} is also a Sylow p-s.g. of $N_G(P)$, but $P \trianglelefteq N_G(P)$ means there is only one Sylow p-s.g. here, hence $P = gPg^{-1}$, hence $P \trianglelefteq N_G(N_G(P))$

Now, $N_G(P)$ is the largest subgroup of G s.t. P is normal in it, hence $\underline{N_G(P) = N_G(N_G(P))}$

(ii) If $K \trianglelefteq G$ and $P \leq K$, show: $KN_G(P) = G$

First, since $P \leq K$, P is also a Sylow p-s.g. of K. Now for $g \in G$, gPg^{-1} is also a Sylow p-s.g. of K since K is closed under conj. All Sylow p-s.g. are conjugate in K, hence $\exists\, k \in K$ such that $gPg^{-1} = kPk^{-1}$, hence for $s \in P$, $\exists\, t \in P$ such that

$gsg^{-1} = ktk^{-1} \Rightarrow g = \underline{kt}\,\underline{k^{-1}gs^{-1}}$

$k \in K, t \in P \leq K$
$\Rightarrow \underline{kt \in K}$

$$k^{-1}gs^{-1}Psg^{-1}k$$
$$= k^{-1}gPg^{-1}k \text{ since } s \in P$$
$$= k^{-1}kPk^{-1}k$$
$$= P \Rightarrow k^{-1}gs^{-1} \in N_G(P)$$

(iii) If no proper s.g. of G is its own normalizer, show that $Z(G) \neq 1$:

Recall $N_G(P) = N_G(N_G(P))$, hence $N_G(P)$ cannot be proper, hence $N_G(P) = G$, hence $P \trianglelefteq G$ $\forall$ Sylow p-s.g's P

Hence G is a direct sum of its Sylow p-subgroups, all of which have nontrivial centers since p-groups.

② Up to isomorphism describe all finitely-generated abelian groups which satisfy all of these:

(1) $G \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^2$

(2) $G \otimes_{\mathbb{Z}} (\mathbb{Z}_7) \cong (\mathbb{Z}_7)^3$

(3) $G \otimes_{\mathbb{Z}} (\mathbb{Z}_p) \cong (\mathbb{Z}_p)^2$ for $p \neq 7$.

$G \cong \mathbb{Z}^m \oplus G_T$

(1): $(\mathbb{Z}^k \oplus G_T) \otimes \mathbb{Q} \cong (\mathbb{Z}^k \otimes \mathbb{Q}) \oplus \cancel{(G_T \otimes \mathbb{Q})}^{\,0}$

$$\cong \mathbb{Q}^m \cong \mathbb{Q}^2 \Rightarrow \underline{m = 2}$$

$G_T \cong \mathbb{Z}_{p_1^{e_1}} \oplus \dots \oplus \mathbb{Z}_{p_k^{e_k}}$

(2): $(\mathbb{Z}^2 \oplus G_T) \otimes_{\mathbb{Z}} \mathbb{Z}_7 \cong (\mathbb{Z}^2 \otimes \mathbb{Z}_7) \oplus (G_T \otimes_{\mathbb{Z}} \mathbb{Z}_7)$

$$\cong \mathbb{Z}_7^{\,2} \oplus (G_T \otimes_{\mathbb{Z}} \mathbb{Z}_7).$$

Hence $G_T \otimes_{\mathbb{Z}} \mathbb{Z}_7 \cong \mathbb{Z}_7$.

$$\Rightarrow (\mathbb{Z}_{p_1^{e_1}} \oplus \dots \oplus \mathbb{Z}_{p_k^{e_k}}) \otimes \mathbb{Z}_7 \cong (\mathbb{Z}_{p_1^{e_1}} \otimes \mathbb{Z}_7) \oplus \dots \oplus (\mathbb{Z}_{p_k^{e_k}} \otimes \mathbb{Z}_7)$$

$$\cong \mathbb{Z}_7$$

$\Rightarrow p_i = 7$ for only one $i = 1, \dots, k$.

i.e. $G_T \cong \mathbb{Z}_{7^{e_1}} \oplus \mathbb{Z}_{p_2^{e_2}} \oplus \dots \oplus \mathbb{Z}_{p_k^{e_k}}$, $\gcd(p_i, 7) = 1$.

(3): $G \otimes_{\mathbb{Z}} (\mathbb{Z}_p) \cong (\mathbb{Z}_p)^2$ for $p \neq 7$.

$(\mathbb{Z}^2 \oplus G_T) \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong (\mathbb{Z}^2 \otimes \mathbb{Z}_p) \oplus (G_T \otimes \mathbb{Z}_p)$

$$\cong \mathbb{Z}_p^{\,2} \oplus (G_T \otimes \mathbb{Z}_p).$$

$\Rightarrow G_T \otimes \mathbb{Z}_p \cong 0$; hence all summands orders are prime to p for all $p \neq 7$, hence only a 7-primary summand:

$$\Rightarrow \boxed{G \cong \mathbb{Z}^2 \oplus \mathbb{Z}_{7^e}}$$

⑤ R commutative ring with 1 and let $r_1, \ldots, r_n \in R$ satisfy $R = Rr_1 + \cdots + Rr_n$. If $M = \{(a_1, \ldots, a_n) \in R^n : a_1 r_1 + \cdots + a_n r_n = 0\}$, show M is projective R-module.

Consider the map $\varphi : R^n \longrightarrow R$

$$(a_1, \ldots, a_n) \mapsto a_1 r_1 + \cdots + a_n r_n.$$

Now, since $R = Rr_1 + \cdots + Rr_n$, $\exists (s_1, \ldots, s_n) \in R^n$ such that $s_1 r_1 + \cdots + s_n r_n = 1$, hence $\varphi(s_1, \ldots, s_n) = 1$, hence φ surjective. Now note that $\ker \varphi = M$ by construction; ~~then we get~~ ~~$R^n/M \cong R$ by First Isom.~~

Now consider $0 \to M \hookrightarrow R^n \xrightarrow{\varphi} R \to 0$

since $\ker\varphi = M$ this sequence is exact, and R is a free R-module, hence it splits and we get

$$R^n \cong M \oplus R, \text{ hence } M \text{ projective.}$$

(6) $K/\mathbb{Q}$ finite galois ext, $Gal(K/\mathbb{Q}) \cong A_4$.
How many subfields does K contain, what are their dimensions over $\mathbb{Q}$, and which are Galois?

Fundamental theorem of Galois Theory tells us that the subgroups of A_4 correspond to the subfields of K.

$|A_4| = 12 = 3 \cdot 2^2$.

There is no order 6 subgroup, but we have subgroups of order 3, 2 and 2^2.

- The order 2^2 sg. is normal (unique since normal Sylow 2 sg.) and $\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This corresponds to the Galois subextension. This is only normal sg., hence only Galois subext. order 4 $\Rightarrow$ index 3 $\Rightarrow$ $[K^{\mathbb{Z}_2 \oplus \mathbb{Z}_2} : \mathbb{Q}] = 3$.
- There are 3 order 2 subgroups given by the double transpositions; therefore index 6, hence $[K^{\mathbb{Z}_2} : \mathbb{Q}] = 6$.
- There are 4 order 3 subgroups given by 3cycles; index 4, hence $[K^{\mathbb{Z}_3} : \mathbb{Q}] = 3$. for each

total of 8 subextensions