1. Galois
2. Sylow
3. Nullstellensatz (varieties) ← very easy
4. Artin – Wedderburn
5.
6.
7. ??

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4 ⓐ ~~Problem~~ show ring simple. Method: (i) Show semisimple
(ii) Apply AW
(iii) Show 1 copy.

$R \cong M_{n_1}(D_1) \times \cdots \times D_{n_k}(D_k)$

- R simple iff $k=1$
- R product of fields iff $n_1 = \cdots = n_k = 1$ & D_i fields
- show R has identity, need to show R product of fields.

> RE $J(R)=0$: If can't show no nilpotents, then work w/ $R/J(R)$ because $J(R/J(R))=0$ and then try to lift, e.g. if there is identity.

Arguments

Def/ R is Artinian semisimple if $J(R)=0$.
↑ Jacobson radical

Tricky part: show $J(R)=0$.
Many times just use $J(R) \subset$ nilradical & show R has no nilpotents.

(Recall: all finite division rings are fields.)

Then/ R artinian + artinian s.s. $\Rightarrow R$ s.s.
↑ usually given or trivial

no nilpotents $\Rightarrow n_1 = \cdots = n_k = 1$. (If R finite $\Rightarrow$ each D_i is finite)
If R is given as f.d. alg over field, then each D_i is a f.g. alg/ field. If $R = \mathbb{R}$, then D_i is $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$. If $\mathbb{R}$ is comm. then no $\mathbb{H}$'s (or some kind of relation)

Lying Over Theorem.

$R \subset S$ integral.

Then every prime of R is the intersection of a prime in S w/ R.

Nöther Normalization

R f.g. F alg.

and a domain.

$F \hookrightarrow R$.

Then there is an intermediate polynomial ring $F \hookrightarrow F[x_1, \ldots, x_n] \hookrightarrow R$

s.t. $F[x_1, \ldots, x_n] \hookrightarrow R$ is integral

A (not necessarily commutative) ring (w/ 1).
'module' means 'left A-module'.

Def/ A module over A is <u>semisimple</u> if it can be decomposed as a direct sum of simple submodules. (Not necessarily a finite #.)

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e.g. A simple module M over a division ring D is isomorphic to D as a D-module over itself.

<u>Proof:</u> Let $m \in M$, $m \neq 0$. $D \xrightarrow{\varphi} M$, $r \mapsto rm$.
The image of φ is M since M is simple.
$rm = 0 \Rightarrow r^{-1}rm = m = r^{-1}0 = 0$. $\Rightarrow\Leftarrow$
So the only r satisfying $rm = 0$ is $r = 0$.
So the map is injective. $\square$.

<u>Fact:</u> Every module over a division ring D is free and all bases have the same cardinality. Hence it makes sense to talk about the dim of a module over a Division ring.

e.g. A module over $\mathbb{Z}$ is simple if and only if it is $\mathbb{Z}/p\mathbb{Z}$.

<u>Proof:</u> simple $\Longleftrightarrow$ $\langle x \rangle = M$, so cyclic w/ prime order. and no subgroups $\square$

<u>Proposition</u> Let (left/right) A be a ring and let M be an A-module.
TFAE (i) M is semisimple. (ii) M is a sum of simple modules (not assuming direct sum). (iii) Every epimorphism $M \to L$ splits. (iv) Every submodule of M is a direct summand.

<u>Cor</u> If M is semisimple, then submodules and quotients of M are semisimple.

Def/ A ring A is <u>semisimple</u> if it is semisimple as a module over itself. 2

Rmk/ • If a ring A is semisimple, then every A-module is semisimple, for it is a quotient of a free A-module. (Direct sums of semisimple modules are of course semisimple.)

• If A is semisimple, then any short exact sequence of A-modules

$$0 \to M \to N \to L \to 0 \text{ splits}$$

since N is semisimple, hence $N \rightleftarrows L$ splits.

• Every A-module M is a direct summand of a free A-module. Indeed, there is a SES

$$0 \to \ker\pi \to F \xrightarrow{\pi} M \to 0$$

where F is a free A-module.

• Consequently, every module over a semisimple ring is projective. Conversely, if every A-module is projective, then every epimorphism $A \to M$ splits, hence A is semisimple.

Summarizing:

Cor/ A ring A is semisimple iff every A-module is projective (iff every A-module is injective).

<u>Matrix Rings</u> Let D be a division ring, let M be a finite dimensional D-module of dim n. Let $\{x_1, \dots, x_n\}$ be a basis for M.

Define a map $\mathrm{Hom}_D(M,M) \longrightarrow M_n(D)^{op}$ by the correspondence $f(X_i) = \sum_{j=1}^{n} a_{ij} X_j$.

Then this map is a ring isomorphism.

$M_n(D)^{op} \cong M_n(D^{op})$ by the map $A \mapsto A^T$.

Rmk/ • $M_n(D)$ is a semisimple ring when D is a division ring. (Both as left and right $M_n(D)$-modules.)

• Since $M_n(D)$ is finite dimensional over D, $M_n(D)$ is Artinian/Nötherian as a ring. Indeed, a $M_n(D)$-submodule is a D-submodule since D acts as the scalar matrices.

• The same proof as for fields shows in fact $M_n(D)$ is a simple ring (for 2-sided ideals).

Def/ Let k be a field and G a group and denote by kG the group ring.

Thm/ (Maschke) If k is a field and G is a finite group s.t. $\mathrm{char}\, k \nmid |G|$. Then kG is semisimple.

Proposition

Let A be a semisimple ring. Then A is a direct sum of finitely many simple submodules (i.e. minimal left ideals). Also, A is (left) Artinian and (left) Nötherian.

DIGRESSION ON THE JACOBSON RADICAL OF A RING.

Def/ Let A be a ring. The annihilator of an A-module M is

$$\mathrm{Ann}_A(M) = \{a \in A : ax = 0 \text{ for all } x \in M\}.$$

If $\mathrm{Ann}_A(M) = \{0\}$, M is called faithful.

Rmk/

- $\mathrm{Ann}_A(M)$ is a 2-sided ideal of A.
- If $I \subset \mathrm{Ann}_A(M)$ is a 2-sided ideal, then M is naturally an A/I-module.
- If $N \subset M$ is a submodule, then $\mathrm{Ann}_A(M) \subset \mathrm{Ann}_A(N)$.
- If N_1 and N_2 are submodules of M, then $\mathrm{Ann}_A(N_1 + N_2) = \mathrm{Ann}_A(N_1) \cap \mathrm{Ann}_A(N_2)$.

Def/ The <u>Jacobson radical</u> rad(A) of a ring A is defined to be

$$\mathrm{rad}(A) = \bigcap_{\substack{M \text{ a simple} \\ \text{left } A\text{-module}}} \mathrm{Ann}_A(M).$$

Rmk/ • rad(A) is an 2-sided ideal.

• Every simple A-module is naturally a simple $A/\mathrm{rad}(A)$-module. (It is clear that the submodules of M as an A-module or an $A/\mathrm{rad}(A)$-module are the same.) So the converse is true as well. Summarizing: $\left\{\begin{matrix}\text{simple}\\ A\text{-modules}\end{matrix}\right\} = \left\{\begin{matrix}\text{simple}\\ A/\mathrm{rad}A\text{-modules}\end{matrix}\right\}$.

• $\mathrm{rad}(A/\mathrm{rad}(A)) = \{0\}$:

equals $\displaystyle\bigcap_{\substack{M \text{ a simple} \\ A\text{-module}}} \mathrm{Ann}_{A/\mathrm{rad}(A)}(M)$.

$$= \bigcap_{\substack{M \text{ a simple} \\ A\text{-module}}} \overline{\mathrm{Ann}_A(M)} = \{0\}.$$

Thm/ rad(A) is independent of whether we consider left or right modules. (see below).

Thm/ $\mathrm{rad}(A) = \bigcap_{\substack{\text{maximal} \\ \text{proper} \\ \text{left ideals } L}} L$

Thm/ $\mathrm{rad}(A) = \{x \in A : 1 - axb \text{ is a unit for all } a, b \in A\}$

$= \{x \in A : 1 - ax \text{ is left invertible for all } a \in A\}$

$= \{x \in A : 1 - xb \text{ is right invertible for all } b \in A\}.$

Cor/ rad(A) is independent of left or right.

Def/ If $L \subset A$ is an additive s.g.

then L^k is $\left\{ \sum_{i=1}^{N} x_1^i \cdots x_k^i : x_j^i \in L \right\}$.

If L is a left/right/2-sided ideal, then the same is true of L^k.

If $L^k = \{0\}$ then L is called nilpotent.

In particular, every elt of L is a nilpotent elt, and we say L is a nil ideal.

(L nilpotent $\overset{\Longrightarrow}{\not\Longleftarrow}$ L nil).

Thm/ (i) rad(A) contains every nil left/right ideal. (ii) If A is left or right Artinian, then rad(A) is nilpotent. Hence in this case rad(A) is the maximal left/right nilpotent ideal. (A nilpotent left/right ideal is a nil left/right ideal.)

Calculation in the Proof:

Let L be nil left ideal. Let $x \in L$. We must show for every $a \in A$ that $1 - ax$ is left invertible. We conclude $L \subset \operatorname{rad} A$.

$y = ax \in L$. Let $y^k = 0$. Then

$$1 = 1 - y^k = (1 + y + \dots + y^{k-1})(1 - y)$$
$$= (1 + y + \dots + y^{k-1})(1 - ax). \checkmark$$

Cor/ If A is (left/right) Artinian, then every (left/right) nil ideal is nilpotent.

(Proof: There is fixed N s.t. $x \in \operatorname{rad}(A) \Rightarrow x^N = 0$.)
Every nil ideal I is contained in $\operatorname{rad}(A)$.
So $I^N = \{0\}$.

END OF DIGRESSION.

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Thm/ Let A be a ring.

Then A is semisimple $\Longleftrightarrow$ A is (left) Artinian and $\mathrm{rad}(A) = 0$.

<u>Proof</u>: Let A be semisimple. So A is Artinian, and $A = \bigoplus_{j=1}^{N} L_j$ is the direct sum of finitely many simple left ideals. For each $j = 1, 2, \ldots, N$, $M_j = \sum_{\substack{i=1 \\ i \neq j}}^{N} L_i$ is a maximal left ideal of A, and $\bigcap_{j=1}^{N} M_j = \{0\}$. Since $\mathrm{rad}(A)$ is the intersection of all proper maximal left ideals, it follows $\mathrm{rad}(A) = 0$.

Conversely, assume A is left Artinian and $\mathrm{rad}(A) = 0$. Consider the family of left ideals L of A s.t. $A = S \oplus L$ where S is a semisimple left module of A (e.g. $S = 0$, $L = A$). Let L be minimal among this family; L exists since A is left Artinian. It will be shown that if $L \neq 0$, then we may write $L = M \oplus L'$

where M is a nonzero minimal left ideal and L' is a left ideal; then

$$A = (S \oplus M) \oplus L'$$

and $S \oplus M$ is semisimple (as M is simple), contradicting the minimality of L.

So let $L \neq 0$. Again using that A is left Artinian let $M \neq 0$ be a minimal left ideal contained in L. Then $M^2 \subset M$ and either $M^2 = 0$ or $M^2 = M$. In the case $M^2 = 0$, M is a nilpotent left ideal, so M is a nil left ideal, so M is contained in $\mathrm{rad}(A) = 0$ (as A is left-Artinian). $\Rightarrow\Leftarrow$. So $M^2 = M$. Thus we may choose $x \in M$ s.t. $Mx \neq \{0\}$. ($x \neq 0$) $Mx \subset M$ so $Mx = M$. So there is $e \in M$ s.t. $ex = x$. So $e^2 x = ex$, so $(e^2 - e)x = 0$. (so $e \neq 0$)
Observe now that $M_1 = \{z \in M : zx = 0\}$ is a left ideal contained in M. Also note $M_1 \neq M$ (as $Mx \neq 0$). Hence $M_1 = 0$.

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Since $e^2 - e \in M_1$,

we gather $e^2 = e$, i.e. e is an idempotent.

Write $1 = e + (1-e)$. For each $a \in L$,

$a = ae + a(1-e)$. So

$L \subset Le + L(1-e)$.

Also, $Le \subset M$ and $e = e^2 \in Le$.

So $Le \neq 0$. So $Le = M$.

Also, $L(1-e) \subset L + Le \subset L$ ($e \in M \subset L$).

So $L \subset Le + L(1-e) = M + L(1-e) \equiv M + L' \subset L$.

So $L = M + L'$.

The sum is direct:

i.e. $M \cap L' = 0$:

Since $M = Le$, an arbitrary elt is ae, $a \in L$.

So assume $ae = b(1-e)$ where $a, b \in L$.

Then $ae = ae^2 = b(1-e)e = b(e-e^2) = 0$.

We conclude $L = 0$,

hence A is semisimple. $\square$

Thm (~~Wedderburn~~) Any semisimple ring is a direct product of finitely many simple Artinian rings.

Rmk/ • The converse is true:
Every direct product of simple Artinian rings is semisimple. Indeed, we will see that every simple Artinian ring is a matrix ring over a division ring.

Prop/ (Schur's lemma) Let A be a ring. Let M & N be simple A-modules.
(i) Any A-module homomorphism $M \to N$ is 0 or an iso.
(ii) $\mathrm{Hom}_A(M, M)$ is a division ring.
Proof: Immediate. $\square$.

Rmk/ • Let M be a left A-module. Set $D = \mathrm{Hom}_A(M, M)$.
Then M is a D-module by setting $\alpha \cdot m := \alpha(m)$ for $\alpha \in D$, $m \in M$.
• For each $a \in A$, we obtain an elt λ_a of $\mathrm{Hom}_D(M, M)$ by setting $\lambda_a(m) = a \cdot m$.
The induced map $A \longrightarrow \mathrm{Hom}_D(M, M)$ is a ring hom.
"This is a natural representation of A in the D-module M."

Thm/ (Rieffel) Let A be simple. Let $M \neq 0$ be a left ideal of A. Set $D = \mathrm{Hom}_A(M, M)$.
Then the representation $A \to \mathrm{Hom}_D(M, M)$ is an isomorphism (of rings)
i.e. every D-module map $M \to M$ is faithfully mult on the left by some $a \in A$.

Cor/ Let A be a simple Artinian ring.
Let M be a minimal left ideal of A.
Set $D = Hom_A(M, M)$. Then

$$A \cong Hom_D(M, M) \cong M_n(D)^{op} \cong M_n(D^{op})$$

where $n = \dim_D M$ (note: D is a division ring by Schur's lemma, since M is a simple A-module).

Conclusion: A is a simple Artinian ring if and only if A is isomorphic to a matrix ring over a division ring. (In a very natural way.)

Proof: Lemma/ Let D be a division ring. Let M be a D-module. Then $Hom_D(M, M)$ is Artinian if and only if $\dim_D M < \infty$.

Proof (Lemma): If $\dim_D M < \infty$, then since D is a division ring we know $Hom_D(M, M) \cong M_n(D^{op})$ which is Artinian by f.d. together w/ the fact that every $M_n(D^{op})$-submodule is a D^{op}-submodule (D^{op}-subspace). ✓ Conversely, assume M has infinite dimension over D. Then there is an infinite ascending chain of D-subspaces $M_1 \subsetneq M_2 \subsetneq \cdots$
Set $L_k = \{f \in Hom_D(M, M) : f(M_k) = 0\}$. Then L_k is

a left ideal of $\mathrm{Hom}_D(M,M)$, and $L_k \supsetneq L_{k+1}$

(since M is a free D-module, there is a D-module map $M \to M$ vanishing on M_k but not on M_{k+1}).

So $\mathrm{Hom}_D(M,M)$ is not Artinian. $\square$

<u>Proof of Corollary</u> : A simple $\overset{\text{Thm (Rieffel)}}{\Longrightarrow}$ $A \cong \mathrm{Hom}_D(M,M)$.

M minimal $\overset{\text{Schur}}{\Longrightarrow}$ $D = \mathrm{Hom}_A(M,M)$ division ring.

$\therefore A$ Art $\Rightarrow \mathrm{Hom}_D(M,M)$ Art $\overset{\text{lemma}}{\Longrightarrow} \mathrm{Dim}_D M = n < \infty$

$\Rightarrow \mathrm{Hom}_D(M,M) \cong M_n(D^{op})$.

$\square$

Rmk/ • D division ring. $M_n(D)$ is semisimple as a left or right module over itself.

So a direct product of matrix rings over division rings is semisimple.

Thm/ (Wedderburn) A a ring.

~~Thm/ A a ring. If A is left/right Artinian, then A is left/right Noetherian.~~

A is left semisimple

$\Longleftrightarrow$

A is right semisimple

$\Longleftrightarrow$

A is a finite direct product of matrix rings $M_n(D)$ over division rings D.

Proof:/ Apply corollary to previous Wedderburn. $\square$

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Thm/ Let A be a ring. If A is left/right Artinian, then A is left/right Noetherian.

~~Assume A is (left) Artinian. We show A is (left) Nöth.~~

Proof: Let $J = \mathrm{rad}(A)$. Since A is Artinian, J is nilpotent. So there is $k \geq 1$ s.t.

$$A \supset J \supset J^2 \supset \cdots \supset J^k = \{0\}.$$

[If $J^1 = \{0\}$, then A is Artinian w/ vanishing Jacobson radical, hence A is semisimple, hence A is Nötherian.]

To show A is Nöth it suffices to show A/J and J^i/J^{i+1} are Nöth. Indeed, then J^k is Nöth and J^{k-1}/J^k is Nöth; hence J^{k-1} is Nöth and J^{k-2}/J^{k-1} is Nöth; ...; hence J is Nöth and A/J is Nöth; hence A is Nöth.

Since A is Art, A/J is Art. Since $\mathrm{rad}(A/J) = 0$, and A/J Art, it follows A/J is semisimple. Hence A/J is Nöth.

$\longrightarrow$

Consider J^i/J^{i+1} as a left A-module. Since J is the Jacobson radical of A, J^i/J^{i+1} is naturally a left A/J-module.

We claim <u>J^i/J^{i+1} is left Art as an A-module.</u>

Indeed, A l.Art$/A$ $\Longrightarrow$ J^i l.Art$/A$ $\Longrightarrow$ J^i/J^{i+1} l.Art$/A$.

So the claim is proved.

Since A/J is semisimple, J^i/J^{i+1} is semisimple as a left A/J-module. Hence $J^i/J^{i+1} = \bigoplus_{k \in K_i} M_k^i$ is the direct sum of simple A/J-modules. Since the submodules of J^i/J^{i+1} are the same whether we view it as an A-module or A/J-module, <u>J^i/J^{i+1} is l.Artinian as an A/J-module.</u>

If the decomposition of J^i/J^{i+1} into simple modules were infinite, then clearly J^i/J^{i+1} would have a nonterminating descending chain. So the sum is finite, and thus J^i/J^{i+1} is l. Nöth as an A/J-module, being a finite sum of <u>simple</u> A/J-modules.

(A finite direct sum of Art/Nöth modules is Art(Nöth.)) $\square$

(Since A/J is l.semisimple, A/J is r.semisimple, so J^i/J^{i+1} is semisimple as a right A/J-module. But we only know J^i/J^{i+1} is Art as a left A/J-module since A is (left) Artinian. So can't conclude r.semisimple breaks into finite pieces.)