Algebra Exam February 2016

Show your work. Be as clear as possible. Do all problems.

1. Let R be a Noetherian commutative ring with 1 and $I \neq 0$ an ideal of R. Show that there exist finitely many nonzero prime ideals P_i of R (not necessarily distinct) so that $\Pi_i P_i \subset I$ (Hint: consider the set of ideals which are not of that form).

2. Describe all groups of order 130: show that every such group is isomorphic to a direct sum of dihedral and cyclic groups of suitable orders.

3. Let $f(x) = x^{12} + 2x^6 - 2x^3 + 2 \in \mathbb{Q}[x]$. Show $f(x)$ is irreducible. Let K be the splitting field of $f(x)$ over $\mathbb{Q}$. Determine whether $\mathrm{Gal}(K/Q)$ is solvable.

4. Determine up to isomorphism the algebra structure of $\mathbb{C}[G]$ where $G = S_3$ is the symmetric group of degree 3 (Recall that $\mathbb{C}[G]$ is the group algebra of G which has basis G and the multiplication comes from the multiplication on G).

5. If F is a field and $n > 1$ show that for any nonconstant $g \in F[x_1, \ldots, x_n]$ the ideal $gF[x_1, \ldots, x_n]$ is not a maximal ideal of $F[x_1, \ldots, x_n]$.

6. Let F be a field and let P be a submodule of $F[x]^n$. Suppose that the quotient module $M := F[x]^n/P$ is Artinian. Show that M is finite dimensional over F.

Spring 2016 Algebra

(1) R Nöth comm ring w/ 1 and

$I \neq 0$ ideal.

Show there are finitely many nonzero prime ideals P_i s.t. $\prod_i P_i \subset I$.

<u>solution</u>

Let $\mathcal{S}$ be the collection of nonzero ideals not containing a finite product of primes.

Suppose $\mathcal{S} \neq \emptyset$. Since R is Nöth let I be maximal in $\mathcal{S}$. we show I is prime, which is a contradiction.

Suppose there is $xy \in I$ but $x \notin I$ and $y \notin I$.

Then $I \subsetneq I + Rx$.

Similarly $I \subsetneq I + Ry$.

Since I is maximal and $I+Rx \neq 0$, $I+Ry \neq 0$ there are primes $P_1, \dots, P_n, Q_1, \dots, Q_m$ s.t.

$$P_1 \cdots P_n \subset I + Rx \quad \text{and} \quad Q_1 \cdots Q_m \subset I + Ry.$$

But then $P_1 \cdots P_n Q_1 \cdots Q_m \subset (I+Rx)(I+Ry) \subset I$.

~~[illegible]~~ $\Rightarrow\Leftarrow$.

Therefore, $\mathcal{I} = \emptyset$. $\square$

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(2) Describe all groups of order $130 = 2 \cdot 5 \cdot 13$. Show these are all direct sums of cyclic and dihedral groups.

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solution

$n_2 = 1, 5, 13, 5 \cdot 13$ $\quad P = \langle x \rangle$ Sylow 2-s.g.

$n_5 = 1, \cancel{2}, \cancel{13}, 2 \cdot 13$ $\quad Q = \langle y \rangle$ Sylow 5-s.g.

$n_{13} = 1, \cancel{2}, \cancel{5}, \cancel{10}$ $\quad R = \langle z \rangle$ Sylow 13-s.g.

In QR, $n_5 = 1, \cancel{13} \Rightarrow QR \cong Q \times R$.

$R \triangleleft G$.

$[G : QR] = 2$

$\Rightarrow QR \triangleleft G$.

$\Rightarrow G \cong QR \rtimes P$.

$$\langle x \rangle \xrightarrow{\varphi} \mathrm{Aut}(Q \times R) \cong C_4 \times C_{12} = \langle \alpha, \beta \rangle.$$

$|\varphi(x)| = 1$ or 2.

so possible φ: $x \mapsto (1,1)$ or $(\alpha^2, 1)$ or $(1, \beta^6)$ or (α^2, β^6).

For the generators α and β we may take

$\alpha : y \mapsto y^2$ and $\beta : z \mapsto z^2$. $\quad \alpha^2 : y \mapsto y^4$. $\quad \beta^6 : z \mapsto z^{12}$.

Each different pair $X \mapsto (\alpha^i, \beta^j)$ $\begin{array}{l} i = 0, 2 \\ j = 0, 6 \end{array}$ gives a different isomorphism class. Besides $C_2 \times C_5 \times C_{13}$ have:

① $\langle x, y, z \mid x^2 = y^5 = z^{13} = 1, yz = zy, \underline{xy = y^4 x,\ xz = zx} \rangle \cong C_{13} \times D_{10}$

② " $\qquad$ " $\qquad \underline{xy = yx,\ xz = z^{12} x} \rangle \cong C_5 \times D_{26}$

③ " $\qquad$ " $\qquad \underline{xy = y^4 x,\ xz = z^{12} x} \rangle \cong D_{130}$

The last isomorphism is because $x(yz) = y^{-1} z^{-1} x = (zy)^{-1} x = (yz)^{-1} x$

and $|yz| = 13 \cdot 5 = 65$ since $yz = zy$. $\square$.

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③ $$f(X) = X^{12} + 2X^6 - 2X^3 + 2 \in \mathbb{Q}[X].$$

Show $f(X)$ is irred. If L is the splitting field of $f(X)$ over $\mathbb{Q}$, determine whether $Gal(K/\mathbb{Q})$ is solvable.

CHAPTER 14

Top-dimensional de Rham cohomology

A corollary of the Stokes Theorem is the following:

COROLLARY 14.1. *Let M be an oriented manifold of dimension m, with no boundary. There is a well-defined map*

$$I\colon H^m_c(M) \to \mathbb{R}$$

wich to the class of $\omega \in \Omega^m_c(M)$ associates its integral $\int_M \omega$.

THEOREM 14.2. *Let M be an oriented manifold of dimension m, with no boundary. Supposed in addition that M is connected. Then the above map $I\colon H^m_c(M) \to \mathbb{R}$ is an isomorphism.*

EXAMPLE 14.3. $H^{n-1}(\mathbb{R}^n - \{0\}) \cong H^{n-1}(S^{n-1}) \cong \mathbb{R}$.

1. A topological application

Let $B^n = \{x \in \mathbb{R}^n ; \|x\| \leqslant 1\}$ be the unit closed ball in $\mathbb{R}^n$. This is a manifold-with-boundary, with boundary the $(n-1)$–sphere S^{n-1}.

PROPOSITION 14.4. *There is no differentiable map $r\colon B^n \to S^{n-1}$ such that $r(x) = x$ for every $x \in S^{n-1}$.*

COROLLARY 14.5 (Brouwer Fixed Point Theorem). *For every differentiable map $f\colon B^n \to B^n$, there exists at least one point $x \in M$ with $f(x) = x$.*

<u>solution</u> $f(X)$ is irred over $\mathbb{Q}$ by Eisenstein.

$y = x^3$. $\quad y^4 + 2y^2 - 2y + 2 = f(y)$.

Since f is irreducible over $\mathbb{Q}$, f is separable. Since f is irreducible and separable, the Galois group of f is contained in S_4 and hence is solvable.

Let $g \equiv f(y) \in \mathbb{Q}[X]$. Then we gather

$$\mathbb{Q} \subset L' \subset \mathbb{Q}(\alpha_1, \dots, \alpha_n) \quad \text{where}$$

L' is the (Galois) splitting field of g and $\mathbb{Q}(\alpha_1, \dots, \alpha_n)$ is a radical ext over $\mathbb{Q}$. Let $\beta_1, \dots, \beta_4$ be the roots of g and let ξ be a primitive third root of unity. Then $\mathbb{Q} \subset L \subset \mathbb{Q}(\alpha_1, \dots, \alpha_n, \beta_1 \xi^{i_1}, \dots, \beta_4 \xi^{i_4})_{i_1, \dots, i_4 = 0,1,2}$ where L is the splitting field of f. Since each $\beta_j \xi^{i_j}$ is a cube root of an elt of $L' \subset \mathbb{Q}(\alpha_1,\dots,\alpha_n)$

and since $\mathbb{Q}(\alpha_1,\dots,\alpha_n)/\mathbb{Q}$ is radical,

we conclude L is contained in the radical

ext $\mathbb{Q}(\alpha_1,\dots,\alpha_n,\beta_j^{ij})/\mathbb{Q}$ hence

the Galois ext $L/\mathbb{Q}$ is solvable. $\square$.

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(4) Determine up to isomorphism the algebra structure of $\mathbb{C}[G]$ where $G = S_3$, viewed as a $\mathbb{C}$-alg.

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solution

By Maschke's thm,

$$\mathbb{C}[G] \cong \prod_{i=1}^{r} M_{n_i}(\mathbb{C})$$ where r is $\#\{\text{irreducible representations of } G\}$

where n_i is the degree of the i^{th} irred representation

S_3 has 3 conjugacy classes $\{1\}$, $\{(12),(13),(23)\}$, $\{(123),(132)\}$ hence 3 irred representations. There are two deg 1 irred reps: the trivial rep and that given by the signature of the permutation.

Since $n_1^2 + n_2^2 + n_3^2 = 6$ we gather

$$\begin{array}{c} n_1^2 + n_2^2 + n_3^2 \\ \shortparallel \\ 1 + 1 + n_3^2 \end{array}$$

the third irred rep has degree 2.

we conclude

$$\mathbb{C}[S_3] \cong \mathbb{C} \times \mathbb{C} \times M_2(\mathbb{C}). \quad \square.$$

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(5) Let K be a field and $n > 1$.

Show for any nonconstant $g \in K[X_1, \dots, X_n]$

the ideal $g K[X_1, \dots, X_n]$ is not a maximal ideal of $K[X_1, \dots, X_n]$.

<u>solution</u>

If K is algebraically closed, then the result follows by the Nullstellensatz which gives every maximal ideal to be of the form $(X_1 - a_1, \dots, X_n - a_n)$ for $a_i \in K$.

Assume k is any field. Let $m \subset k$ be a maximal ideal. Let $M := \bar{k}[X_1,\dots,X_n]\,m$ be the push-forward ideal of m in $\bar{k}[X_1,\dots,X_n]$ where $\bar{k}$ is the algebraic closure of k.

Let $M \subset (X_1-a_1,\dots,X_n-a_n)$ for some $a_i \in \bar{k}$.

The ideal $A := (X_1-a_1,\dots,X_n-a_n) \cap k[X_1,\dots,X_n]$ of $k[X_1,\dots,X_n]$ contains m, hence $A = m$ by maximality. ~~Let $M \subset (X_1-a_1,\dots,X_n-a_n)$ for some $a_i \in \bar{k}$.~~

Let $m_i(X) \in k[X]$ be the minimal poly of a_i, so $X - a_i$ divides $m_i(X_i)$ in $\bar{k}[X_1,\dots,X_n]$. So $m_i(X_i) \in A = m$.

If m is principal, $m = (f)$ for some irreducible $f \in k[X_1,\dots,X_n]$. So f divides $m_i(X_i)$ in $k[X_1,\dots,X_n]$. Since $m_i(X_i)$ is irreducible in $k[X_i]$, it is irreducible in $k[X_1,\dots,X_n]$

(since $\deg_i(fg) = \deg_i(f)\deg_i(g)$ for all $f, g \in k[X_1, \dots, X_n]$ and $i = 1, \dots, n$). So f and $m_i(X_i)$ are associates, since f is not a unit. Since $n \geq 2$ this is a contradiction, because it says f is of deg ≥ 1 in $k[X_1] \cap \dots \cap k[X_n] = k$.

We conclude m is not principal. $\square$

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(6) Let F be a field, and let P be an $F[x]$-submodule of $F[x]^n$. Suppose the quotient $F[x]$-module $M := F[x]^n / P$ is Artinian. Show M is f.d. over F.

M is a f.g. module over a PID, so $M \cong F[x]^r \oplus F[x]/(f_1) \oplus \dots \oplus F[x]/(f_n)$. $F[X]^r$ is not Artinian since there is a non-terminating descending chain $(X, 0, \dots, 0) \supset (X^2, 0, \dots, 0) \supset \dots$. So there is no free part. Since each $F[X]/f_i$ is f.d./F we are done. $\square$