Spring 2025 Solutions

Problem 1

Let $u \not\equiv 0$ be a $C^2(\mathbb{R}^n)$ function ($n > 1$). Define

$$m_x(r) = r^{1-n} \int_{\partial B(x,r)} u(y) dS(y)$$

(a) Show that

$$\frac{dm_x(r)}{dr} = r^{1-n} \int_{B(x,r)} \Delta u(y) dy$$

(b) Let u solve $-\Delta u = \phi(u)$. For some continuous function ϕ. Assume that $u(x) \geq 1$ for all $x \in \mathbb{R}^n$, and that $\phi(\xi) \geq 0$ for all $\xi \geq 1$. Using the result from part (a) above, prove that if $u(x_0) = 1$ for some $x_0 \in \mathbb{R}^n$, then $u(x) \equiv 1$ for all $x \in \mathbb{R}^n$.

(a)

proof:
Reparametrize the sphere with $y = x + r\omega$, with $\omega \in S^{n-1}$, then $dS(y) = r^{n-1} d\sigma(\omega)$ (where $d\sigma$ is the surface measure on S^{n-1}, and $\nu(y) = \omega$ is the outward unit normal on $\partial B(x,r)$). Thus

$$m_x(r) = r^{1-n} \int_{\partial B(x,r)} u(y) dS = \int_{S^{n-1}} u(x + r\omega) d\sigma(\omega)$$

Furthermore, note that

$$\frac{d}{dr} m_x(r) = \int_{S^{n-1}} \frac{d}{dr} u(x + r\omega) d\sigma(\omega)$$

Now we compute $\frac{d}{dr} u(x + r\omega)$

$$\frac{d}{dr} u(x + r\omega) = \nabla u(x + r\omega) \cdot \omega$$

In case the above is not clear, when we differentiate the above, we are differentiating the composition: $r \mapsto u(x + r\omega)$. This is not a partial derivative of u with respect to a coordinate of u, but rather the derivative along a curve in $\mathbb{R}^n$. Thus by the chain rule

$$\begin{aligned}\frac{d}{dr} u(x + r\omega) &= \sum_{i=1}^{n} \frac{\partial}{\partial y_i} u(x + r\omega) \cdot \frac{d}{dr}(x_i + r\omega_i) = \sum_{i=1}^{n} u_{y_i}(x + r\omega)\omega_i \\ &= \nabla u(x + r\omega) \cdot \omega\end{aligned}$$

As a result

$$\begin{aligned}\frac{d}{dr} m_x(r) &= \int_{S^{n-1}} \nabla u(x + r\omega) \cdot \omega d\sigma(\omega) = r^{1-n} \int_{\partial B(x,r)} \nabla u(y) \cdot \nu(y) dS(y) \\ &= r^{1-n} \int_{\partial B(x,r)} \partial_\nu(y) dS(y)\end{aligned}$$

Then by

> Gauss-Green
>
> **Evans p.711**
>
> (i) Suppose $u \in C^1(\overline{U})$. Then
>
> $$\int_U u_{x_i} \, dx = \int_{\partial U} u\nu^i \, dS \quad (i = 1, \ldots, n).$$
>
> (ii) - also called the divergence theorem

$$\int_U \operatorname{div} u \, dx = \int_{\partial U} u \cdot \nu \, dS$$

for each vector field $u \in C^1(\overline{U}; \mathbb{R}^n)$.

#PDE

(divergence theorem)

$$\frac{d}{dr} m_x(r) = r^{1-n} \int_{\partial_{B(x,r)}} \partial_\nu u dS = r^{1-n} \int_{B(x,r)} \operatorname{div}(\nabla u) dy = r^{1-n} \int_{B(x,r)} \Delta u dy$$

■

(b)

proof:
At x_0, we have $\Delta u = -\phi(u) \leq 0$, since $u \geq 1$ and $\phi \geq 0$ on $[1, \infty)$. Plugging this into part (a)'s result

$$\frac{d}{dr} m_{x_0}(r) = r^{1-n} \int_{B(x_0,r)} \Delta u = -r^{1-n} \int_{B(x_0,r)} \phi(u) \leq 0$$

Recall that

$$m_{x_0}(r) = r^{1-n} \int_{\partial B(x_0,r)} u dS = n\alpha(n) ⨍_{\partial B(x_0,r)} u dS$$

Thus, since $u \geq 1$ everywhere we have

$$m_{x_0}(r) \geq n\alpha(n)$$

for all $r > 0$. Then, by continuity, $\lim_{r \downarrow 0} m_{x_0}(r) = n\alpha_n u(x_0) = n\alpha(n)$. Thus

$$m_{x_0}(r) \equiv n\alpha(n) \quad \text{for all } r > 0$$

As a result

$$\frac{d}{dr} m_{x_0}(r) = 0 = -r^{1-n} \int_{B(x_0,r)} \phi(u(y)) dy$$

for all $r > 0$. Hence

$$\int_{B(x_0,r)} \phi(u(y)) dy = 0 \quad \text{for all } r > 0$$
$$\implies \phi(u(y)) = 0 \quad \text{for all } y \in \mathbb{R}^n$$

Thus $\Delta u \equiv 0$ and we have that u is harmonic on $\mathbb{R}^n$. Since $u \geq 1$ everywhere and $u(x_0) = 1$, x_0 is a global minimum of the function and by the maximum principle we have that $u \equiv 1$ on $\mathbb{R}^n$. ■

Problem 2

Use the method of characteristics to solve the following partial differential equation:

$$\begin{aligned} \partial_t u - u \partial_x u &= 3u, \quad x \in \mathbb{R}, \quad t > 0, \\ u(0, x) &= u_0(x), \quad x \in \mathbb{R} \end{aligned}$$

proof:
Using the method of characteristics

$$\frac{dx}{ds} = -z, \quad \frac{dt}{ds} = 1, \quad \frac{dz}{ds} = 3z$$

With initial data

$$x(0, r) = r, \quad t(0, r) = 0, \quad z(0, r) = u_0(r)$$

Thus

$$z = e^{3s}u_0(r), \quad t = s, \quad x = -\frac{z}{3} + \frac{u_0(r)}{3} + r$$

In order to express z in terms of x and t we solve for r to obtain

$$r = x + \frac{z}{3} - \frac{z}{3e^{3s}}$$

Thus

$$u(x,t) = e^{3t}u_0\left(x + u(x,t)\left(1 - \frac{e^{-3t}}{3}\right)\right)$$

■

Problem 3

3. Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain with smooth boundary. Assume that $u(t,x) \geq 0$ is a smooth function solving

$$\begin{cases} \partial_t u - \Delta u = -u^4 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

with $u|_{t=0} = u_0$. Here ν is the unit normal to the boundary. Let

$$E(t) = \int_\Omega u^2(t,x)dx.$$

Show that there exists a constant $C > 0$ such that for each $t > 0$

$$E(t) \leq \frac{1}{(E^{-\frac{3}{2}}(0) + Ct)^{\frac{2}{3}}}$$

proof:
We use the energy identity to take

$$\frac{d}{dt}E(t) = 2\int_\Omega uu_t = 2\int_\Omega u(\Delta u - u^4) = 2\int_\Omega u\Delta u - 2\int_\Omega u^5$$

by

> Green's Formulas
>
> Let $u, v \in C^2(\bar{U})$
> (i)
>
> $$\int_U \Delta u \, dx = \int_{\partial U} \frac{\partial u}{\partial \nu} dS$$
>
> (ii)
>
> $$\int_U \nabla v \cdot \nabla u \, dx = -\int_U u\Delta v \, dx + \int_{\partial U} u\frac{\partial v}{\partial \nu} dS$$
>
> (iii)
>
> $$\int_U u\Delta v - v\Delta u \, dx = \int_{\partial U} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} dS$$
>
> #PDE

we have

$$\int_\Omega u\Delta u = \cancelto{0}{\int_{\partial\Omega} u\frac{du}{d\nu} dS} - \int_\Omega |\nabla u|^2$$

Thus

$$E'(t) = -2\int_\Omega |\nabla u|^2 - 2\int_\Omega u^5 \le -2\int_\Omega u^5$$

Using

> Holder's Inequality
>
> Let $1 \le p, q \le \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then if $f \in L^p(U)$ and $g \in L^q(U)$
>
> $$\int_U |fg|\ dx \le \|f\|_{L^p(U)} \cdot \|g\|_{L^q(U)}$$

we have

$$E(t) = \int_\Omega u^2 = \int_\Omega u^2 \cdot 1 \le \left(\int_\Omega u^5\right)^{2/5} \left(\int_\Omega 1^{5/3}\right) = \left(\int_\Omega u^5\right)^{2/5} |\Omega|^{3/5}$$

As a result $E(t)^{5/2}|\Omega|^{-3/2} \le \int_\Omega u^5 = -\frac{1}{2}E'(t)$, thus

$$E'(t) \le -\frac{2}{|\Omega|^{3/2}}E(t)^{5/2}$$

Lastly, we differentiate $E(t)^{-3/2}$ to obtain

$$\begin{aligned}
\frac{d}{dt}E(t)^{-3/2} &= -\frac{3}{2}E(t)^{-5/2}E'(t) \ge \frac{3}{|\Omega|^{3/2}} \\
\int_0^t \frac{d}{dt}E(t)^{-3/2} &= E(t)^{-3/2} - E(0)^{-3/2} \ge \frac{3}{|\Omega|^{3/2}} \\
&\implies E(t)^{-3/2} \ge E(0)^{-3/2} + \frac{3}{|\Omega|^{3/2}} \\
&\implies E(t) \le \frac{1}{(E(0)^{-3/2} + Ct)^{2/3}}
\end{aligned}$$

where $C = \frac{3}{|\Omega|^{2/3}}$, which is the desired inequality. ■