# Spring 2024 Solutions

# Problem 1

Find a classical solution $u(x, y)$ of the equation

$$(\partial_x u)^2 - x^2 = (\partial_y u)^2 - y^2$$

in $\mathbb{R}^2$ satisfying the boundary condition $u(y, y) = y^2$

proof:
Setup via method of characteristics. Parameterize so that

- $x = x(s)$
- $y = y(s)$
- $z = u(x(s), y(s))$
- $p = u_x(x(s), y(s))$
- $q = u_y(x(s), y(s))$
  Then

$$F(x, y, u, p, q) = p^2(s) - q^2(s) - x(s)^2 + y(s)^2$$

Thus,

$$\begin{cases} \frac{dx}{ds} = 2p(s) \\ \frac{dy}{ds} = -2q(s) \\ \frac{dz}{ds} = 2p(s)^2 - 2q(s)^2 \\ \frac{dp}{ds} = 2x(s) \\ \frac{dq}{ds} = -2y(s) \end{cases}$$

Looking at our structure:
- $\frac{dp}{ds} = 2x(s)$ and $\frac{dx}{ds} = 2p(s)$ are coupled
- Similarly $\frac{dq}{ds} = -2y(s)$ and $\frac{dy}{ds} = -2q(s)$
  Thus both $(x, p)$ and $(y, q)$ form *second-order systems.* In fact, we can differentiate the equations to get second-order ODES

So we have

$$\frac{d^2x}{ds^2} = 2\frac{dp}{ds} = 4x(s)$$

And

$$\frac{d^2y}{ds^2} = -2\frac{dq}{ds} = 4y(s)$$

Thus we have

$$\frac{d^2x}{ds^2} - 4x = 0 \Rightarrow x(s) = c_1e^{2s} + c_2e^{-2s}$$

and similarly

$$\frac{d^2y}{ds^2} - 4y = 0 \Rightarrow y(s) = c_3e^{2s} + c_4e^{-2s}$$

From our initial conditions we have

$$x(0) = y, \quad y(0) = y, \quad z(0) = y^2$$

Thus

$$x(0) = c_1e^0 + c_2e^0 = c_1 + c_2 = r$$

and

$$y(0) = c_3e^0 + c_4e^0 = c_3 + c_4 = r$$

We choose $c_1 = c_4 = r$ and $c_2 = c_3 = 0$. As a result we have

$$x(s) = re^{2s} \quad y(s) = re^{-2s}$$

Now we can plug in

$$\frac{dp}{ds} = 2re^{2s}, \quad \frac{dq}{ds} = -2re^{-2s} \quad \Rightarrow \quad p = re^{2s}, \quad q = -re^{-2s}$$

Hence

$$\frac{dz}{ds} = 2r^2e^{4s} - 2r^2e^{-4s} \Rightarrow z = \frac{r^2}{2}(e^{4s} + e^{-4s}) = \frac{x^2}{2} + \frac{y^2}{2}$$

■

# Problem 2

Let $U$ be open and bounded in $\mathbb{R}^n$. Suppose $u$ and $\Delta u$ are continuous in $U$, and $u$ satisfies that $\begin{cases} \Delta u = u^4 & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$

(1) If $u \geq 0$ in $\overline{U}$, prove that $u \equiv 0$

(2) If the condition $u \geq 0$ in $\overline{U}$ is absent, what can you say about $u(x)$?

## (1)

proof:

By

> Green's Formulas
>
> Let $u, v \in C^2(\bar{U})$
>
> (i)
>
> $$\int_U \Delta u \, dx = \int_{\partial U} \frac{\partial u}{\partial \nu} dS$$
>
> (ii)
>
> $$\int_U \nabla v \cdot \nabla u \, dx = -\int_U u \Delta v \, dx + \int_{\partial U} u \frac{\partial v}{\partial \nu} dS$$
>
> (iii)
>
> $$\int_U u\Delta v - v\Delta u \, dx = \int_{\partial U} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} dS$$
>
> #PDE

$$\int_U u^5 \, dx = \int_U u\Delta u \, dx = -\int_U \nabla u \cdot \nabla u + \int_{\partial U} u\frac{du}{d\nu} dS = -\int_U \nabla u \cdot \nabla u$$

With the last equality following since $u = 0$ on $\partial U$

If $u \geq 0$ in $\overline{U}$ then $u^5 \geq 0$ in $\overline{U}$. Thus

$$0 \leq \int_U u^5 \, dx = -\int_U |Du|^2$$

The only way for the negative integral of a positive function to be greater than or equal to 0 is if it itself is equal to 0. Thus $Du = 0 \Rightarrow u \equiv 0$. ■

You could also use the Weak Maximum Principle taking $L = -\Delta$

## (2)

If the condition $u \geq 0$ in $\overline{U}$ is absent, what can you say about $u(x)$?

proof:
We can only say $u \leq 0$. ■

# Problem 3

Let $P, Q, R, S \in \mathbb{R} \times \mathbb{R}^+$ be consecutive vertices of a rectangle with sides parallel to the lines $x + t = 0$ and $x - t = 0$, respectively.

(1) Let $u = u(x,t) \in C^2(\mathbb{R} \times \mathbb{R}^+)$ be a solution to the wave equation in $\mathbb{R} \times \mathbb{R}^+$, show that

$$u(P) + u(R) = u(Q) + u(S)$$

(2) Let $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ be functions. Find a solution to the wave equation in $\mathbb{R}^2$ such that

$$u(t-1,t) = \alpha(t), \quad u(5-t,t) = \beta(t)$$

## (1)

proof:
By

> Wave Equation General Form for R1
>
> The solution to the wave equation
>
> $$u_{tt} - u_{xx} = 0$$
>
> has the form
>
> $$u(x,t) = F(x+t) + G(x-t)$$
>
> #PDE

we know that any solution to the wave equation in $\mathbb{R}^1$ has the form

$$u = F(x-t) + G(x+t)$$

Take points

$$S = (a,b), P = (a-m, b+m), R = (a+c, b+c), Q = (a-m+c, b+m+c)$$

Thus

$$u(P) + u(R) = F(b-a+2m) + G(a+b) + F(b-a) + G(a+b+2c)$$
$$u(Q) + u(S) = F(a-b+2m) + G(a+b+2c) + F(b-a) + G(a+b)$$

Thus the two are equal
■

## (2)

proof:
Observe once again that

$$u(x,t) = F(x-t) + G(x+t)$$

Then we have

$$u(t-1,t) = F(-1) + G(2t-1) = \alpha(t)$$
$$u(5-t,t) = F(5-2t) + G(5) = \beta(t)$$

Let $s = 2t-1 \Rightarrow t = \frac{s+1}{2}$ then

$$u\left(\frac{s-1}{2}, \frac{s+1}{2}\right) = F(-1) + G(s) = \alpha\left(\frac{s+1}{2}\right) \Rightarrow$$
$$G(s) = \alpha\left(\frac{s+1}{2}\right) - F(-1)$$

Now let $r = 5-2t \Rightarrow t = \frac{5-r}{2}$ then

$$F(r) = \beta\left(\frac{5-r}{2}\right) - G(5)$$

Observe that $G(5) + F(-1) = \alpha(3) = \beta(3)$. Thus

$$u(x,t) = F(x-t) + G(x+t) =$$
$$\beta\left(\frac{5+t-x}{2}\right) - G(5) + \alpha\left(\frac{x+t+1}{2}\right) - F(-1)$$
$$= \beta\left(\frac{5+t-x}{2}\right) + \alpha\left(\frac{x+t+1}{2}\right) - \alpha(3)$$

■