# Spring 2023 Solutions

# Problem 1

Consider the equation

$$u_x^2(x,y) + 2u_y^2(x,y) = x^2 + 2y^2$$

(a) Find at least one classical solution to this equation in $\mathbb{R}^2$ such that $u(x,x) = x^2$

(b) Is the problem from (a) uniquely solvable?

## (a)

proof:
We first set the equation up in the following form to solve via the method of characteristics

$$F(x,y,z,p,q) = u_x^2(x,y) + 2u_y^2(x,y) - x^2 - 2y^2 = p^2 + 2q^2 - x^2 - 2y^2$$

Now we have that

$$\frac{dx}{ds} = F_p, \quad \frac{dy}{ds} = F_q, \quad \frac{dz}{ds} = pF_p + qF_q,$$
$$\frac{dp}{ds} = -F_x - pF_z, \quad \frac{dq}{ds} = -F_y - qF_z$$

Thus

$$\frac{dx}{ds} = 2p, \frac{dy}{ds} = 4q, \frac{dz}{ds} = 2p^2 + 4q^2, \frac{dp}{ds} = 2x, \frac{dq}{ds} = 4y$$

With initial data $x(r,0) = r, y(r,0) = r, z(r,0) = \phi$

Now, we can create a second order differential equation out of $x, p$ and $y, q$ in the following manner

$$\frac{d^2x}{ds^2} = 2\frac{dp}{ds} = 4x, \quad \frac{d^2y}{ds^2} = 4\frac{dq}{ds} = 16y$$

Thus

$$x = c_1e^{2s} + c_2e^{-2s}, \quad y = c_3e^{4s} + c_4e^{-4s}$$

Now based on our initial data we have

$$x(r,0) = c_1 + c_2 = r, \quad y(r,0) = c_3 + c_4 = r$$

Now we make a selection of $c_1 = c_3 = r$ and $c_2 = c_4 = 0$.

Thus $x(r,s) = re^{2s}$ and $y(r,s) = re^{4s}$

In order to find admissible initial data we set $p(r,0) = \psi_1$ and $q(r,0) = \psi_2$ then

$$F(\gamma_1, \gamma_2, \phi(r), \psi_1, \psi_2) = \psi_1^2 + 2\psi_2^2 - r^2 - 2r^2 = 0 \Rightarrow \psi_1^2 + 2\psi_2^2 = 3r^2$$

and

$$\phi'(r) = \gamma_1'(r)\psi_1(r) + \gamma_2'(r)\psi_2(r) = \psi_1(r) + \psi_2(r) = 2r$$

Choosing $\psi_1(r) = r, \psi_2(r) = r$ solves this.

Now observe that

$$\frac{dp}{ds} = 2re^{2s} \Rightarrow p = re^{2s} + p_0 \quad , \frac{dq}{ds} = 4re^{4s} \Rightarrow q = re^{4s} + q_0$$

Now using our initial data

$$p(r,0) = r + p_0 = r \Rightarrow p(r,s) = re^{2s} \quad q(r,0) = r + q_0 = r \Rightarrow q(r,s) = re^{4s}$$

Then

$$\frac{dz}{ds} = 2p^2 + 4q^2 = 2r^2e^{4s} + 4r^2e^{8s} \Rightarrow z = \frac{r^2e^{4s}}{2} + \frac{r^2e^{8s}}{2} + z_0$$

Plugging in initial data gives us $z(r,0) = \frac{r^2}{2} + \frac{r^2}{2} + z_0 = r^2 \Rightarrow z_0 = 0$ so $z = \frac{r^2e^{4s}+r^2e^{8s}}{2}$

Notice that then

$$z(r,s) = u(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$$

Which we can easily verify is a solution.

■

## (b)

Put the Cauchy data on the line $\Gamma = \{(x,y) : y = x\}$ with $\gamma(r) = (r,r)$ and $u_0(r) = r^3$. For a first order equation $F(x,y,u,p,q) = 0$ admissible initial slopes $(p,q) = (u_x, u_y)$ along $\Gamma$ must satisfy the compatibility conditions

$$F(\gamma(r), u_0(r), p, q) = 0$$
$$u_0'(r) = p\gamma_1'(r) + q\gamma_2'(r)$$

Here $F = p^2 + 2q^2 - x^2 - 2y^2$ and $\gamma'(r) = (1,1)$ and $u_0'(r) = 2r$. At the point $(x,y) = (r,r)$ these become

$$p^2 + 2q^2 = 3r^2$$
$$p + q = 2r$$

Thus we have $3q^2 - 4rq + r^2 = 0$ whose roots are

$$q = r \quad \text{or } q = \frac{r}{3}, \qquad p = r \quad \text{or} \quad \frac{5r}{3}$$

Thus for every $r \neq 0$ there are two distinct pairs

$$(p,q) = (r,r) \text{ and } (p,q) = \left(\frac{5r}{3}, \frac{r}{3}\right)$$

Thus the problem is not uniquely solvable. ■

# Problem 2

Let $u$ be harmonic in the domain $U = B(0,4)$ in $\mathbb{R}^2$

(a) Show that if $u$ is bounded then

$$\sup(4 - |x|) \cdot |\nabla u(x)| < \infty$$

(b) Give an example of $u$ that is harmonic in $U$ and unbounded, but still satisfies the claim from $a$

## (a)

Since $u$ is bounded we have that

$$u(x) \leq M \text{ for all } x \in U$$

Then since $u$ is harmonic, by the

> Mean Value Formula

> **Evans p.25:**
> If $u \in C^2(U)$ is harmonic, then
> $$u(x) = ⨍_{\partial B(x,r)} u(y) dS(y) = ⨍_{B(x,r)} u\, dy$$
> for each ball $B(x,r) \subset U$
>
> #PDE

we have then

$$u(x) = ⨍_{\partial B(x,r)} u(y) dS(y) = ⨍_{B(x,r)} u\, dy$$

For each ball $B(x,r) \subset B(0,4)$. Recall that in $\mathbb{R}^n$

$$⨍_{\partial B(x,r)} u(y) dS(y) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} u\, dS(y)$$

and

$$⨍_{B(x,r)} u\, dy = \frac{1}{\alpha(n)r^n} \int_{B(x,r)} u\, dy$$

In this case (for $\mathbb{R}^2$) we have that

> alpha(2)
> $$\alpha(2) = \pi$$
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$= \pi$ thus

$$u(x) = \frac{1}{\pi r^2} \int_{B(x,r)} u\, dy = \frac{1}{2\pi r} \int_{\partial B(x,r)} u\, dy$$

For each ball $B(x,r) \subset B(0,4)$. We now aim to calculate

$$|\nabla u| = |(u_{x_1}, u_{x_2})| = \sqrt{u_{x_1}^2 + u_{x_2}^2}$$

Since $u$ is harmonic, and

> Derivatives of Harmonic Functions are Harmonic

, thus again by the

> Mean Value Formula
>
> **Evans p.25:**
> If $u \in C^2(U)$ is harmonic, then
> $$u(x) = ⨍_{\partial B(x,r)} u(y) dS(y) = ⨍_{B(x,r)} u\, dy$$
> for each ball $B(x,r) \subset U$
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> #PDE

now applied to $u_{x_i}$since $u_{x_i}$ itself is harmonic

$$u_{x_i}(x) = \frac{1}{\pi r^2} \int_{B(x,r)} u_{x_i}\, dy = \frac{1}{\pi r^2} \int_{\partial B(x,r)} u_{x_i}\, dS$$

For each ball $B(x,r) \subset B(0,4)$.

Then by

> Gauss-Green
>
> **Evans p.711**

(i) Suppose $u \in C^1(\overline{U})$. Then

$$\int_U u_{x_i}\, dx = \int_{\partial U} u\nu^i \, dS \quad (i = 1, \ldots, n).$$

(ii) - also called the divergence theorem

$$\int_U \text{div } u \, dx = \int_{\partial U} u \cdot \nu \, dS$$

for each vector field $u \in C^1(\overline{U}; \mathbb{R}^n)$.

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we have

$$\frac{1}{\pi r^2} \int_{B(x,r)} u_{x_i}\, dy = \frac{1}{\pi r^2} \int_{\partial B(x,r)} u\nu^i \, dS \quad (i = 1, \ldots, n).$$

Next, observe

$$|u_{x_i}(x)| = \left| \frac{1}{\pi r^2} \int_{\partial B(x,r)} u_{x_i} \, dS \right| \leq \left( \frac{1}{\pi r^2} \right) \int_{\partial B(x,r)} |u| \cdot \left|\nu^i\right| dS$$

Since $\nu$ is the

Unit Normal (nu)

**Evans p.710**

(i) If $\partial U$ is $C^1$, then along $\partial U$ is defined the *outward pointing unit normal vector field*

$$\nu = (\nu^1, \ldots, \nu^n)$$

The *unit normal at any point* $x^0 \in \partial U$ is $\nu(x^0) = \nu = (\nu_1, \ldots, \nu_n)$

(ii) Let $u \in C^1(\overline{U})$. We call

$$\frac{\partial u}{\partial \nu} := \nu \cdot Du$$

The *(outward)* normal derivative of $u$.

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, we have that

$$|\nu| = 1 \Rightarrow |\nu^i| \leq 1$$

for each $i$. Thus

$$|u_{x_i}| \leq \left( \frac{1}{\pi r^2} \right) \int_{\partial B(x,r)} |u| \cdot \left|\nu^i\right| dS \leq \left( \frac{1}{\pi r^2} \right) \int_{\partial B(x,r)} |u| \, dS$$

Then since

Lebesgue Integral is Bounded by Measure times Sup

For a nonnegative measurable function $f$, and a measure $\mu$

$$\int_X f d\mu \leq \sup_X f \cdot \mu(X)$$

Thus, for any function $f$ and a measure $\mu$ we have

$$|\int_X f d\mu| \leq \int_X |f| d\mu \leq \sup_X |f| \cdot \mu(X)$$

we have that

$$|u_{x_i}| \leq \left( \frac{1}{\pi r^2} \right) \int_{\partial B(x,r)} |u| \, dS \leq \frac{M}{\pi r^2} \cdot n\alpha(n)^{n-1} = \frac{M}{\pi r^2} \cdot 2\pi r = \frac{2M}{r}$$

For each ball $B(x,r) \subset B(0,4)$.

Now, as stated previously

$$|\nabla u| = |(u_{x_1}, u_{x_2})| = \sqrt{u_{x_1}^2 + u_{x_2}^2}$$

so

$$|\nabla u| \leq \sqrt{\frac{4M^2}{r^2} + \frac{4M^2}{r^2}} = \frac{2M\sqrt{2}}{r}$$

Next, we take the radius $r = 4 - |x|$. Then

$$|\nabla u| \leq \left(\frac{2M\sqrt{2}}{4 - |x|}\right)$$

Thus

$$\sup\left((4 - |x|)\left(\nabla u(x)\right)\right) \leq 2M\sqrt{2}$$

■

## (b)

proof:
We know that $\log(|x|)$ is harmonic for $x \in \mathbb{R}^2$, thus we re-center the function so that the singularity is on the boundary of the open ball $B(0, 4)$. Thus take

$$u(x, y) = \log(\sqrt{(x-4)^2 + y^2})$$

Which is still harmonic in the open ball $B(0, 4)$ (but it blows up near the point (4,0) on the boundary). So this function is harmonic and unbounded.

■

# Problem 3

Suppose $u$ is a $C^2$ function satisfying

$$\begin{cases} u_{tt} = \Delta u, & \text{in } \mathbb{R}^n \times \mathbb{R}^+, \\ u(x,0) = g(x), \\ u_t(x,0) = h(x) \end{cases}$$

Fix $T > 0$ and consider the set

$$K_T := \{(x,t) \ : \ 0 \leq t \leq T, \ |x| \leq T - t\}$$

Prove that if $g(x) = h(x) = 0$ for all $x \in B(0,T)$, then $u(x,t) = 0$ for all $(x,t) \in K_T$

proof:
This is exactly the same as

> Finite Propagation Speed
>
> **Evans p.84**
>
> Let
>
> $$K(x_0, t_0) := \{(x,t) \ : \ 0 \leq t \leq t_0, \ |x - x_0| \leq t_0 - t\}$$
>
> If $u \equiv u_t \equiv 0$ on $B(x_0, t_0) \times \{t = 0\}$ then $u \equiv 0$ *within* the cone $K(x_0, t_0)$
>
> proof:
> Define the Local Energy

$$e(t) := \frac{1}{2}\int_{B(x_0,t_0-t)} u_t^2(x,t) + |Du(x,t)^2|\, dx$$

with $0 \le t \le t_0$.

By the Differentiation Formula for Moving Regions we have that for smooth $u$ then

$$2\frac{d}{dt}e(t) =$$
$$\frac{d}{dt}\int_{\partial B(x_0,t_0-t)} (u_t^2 + |Du^2|)\mathbf{v}\cdot\nu\, dS + \int_{B(x_0,t_0-t)} \frac{d}{dt}(u_t^2 \cdot |Du|^2)dx$$

Our velocity of the moving boundary here is $\nu\cdot(t_0-t)' = -\nu$ (thus $\mathbf{v}\cdot\nu = -\nu\cdot\nu = -1$. As a result we have

$$\dot{e}(t) = \int_{B(x_0,t_0-t)} u_t u_{tt} + Du\cdot Du_t\, dx - \frac{1}{2}\int_{\partial B(x_0,t-t_0)} u_t^2 + |Du^2|\, dS$$

Next, by Green's Formulas we have

$$\int_{B(x_0,t_0-t)} \nabla u\cdot\nabla u_t\, dx = -\int_{B(x_0,t_0-t)} u_t\Delta u\, dx + \int_{\partial B(x_0,t_0-t)} u_t\frac{\partial u}{\partial\nu}dS$$

Thus

$$\dot{e}(t) = \int_{B(x_0,t_0-t)} u_t(u_{tt}-\Delta u)dx - \frac{1}{2}\int_{\partial B(x_0,t-t_0)} u_t^2 + |Du^2|dS$$
$$+\int_{\partial B(x_0,t_0-t)} u_t\frac{\partial u}{\partial\nu}dS$$

Since $u_t \equiv 0$ on $B(x_0,t_0-t)$ then

$$\begin{aligned}\dot{e}(t) &= \int_{\partial B(x_0,t_0-t)} u_t\frac{\partial u}{\partial\nu}dS - \frac{1}{2}\int_{\partial B(x_0,t-t_0)} u_t^2 + |Du^2|dS\\ &= \int_{\partial B(x_0,t_0-t)} u_t\frac{\partial u}{\partial\nu}dS - \frac{1}{2}(u_t^2) - \frac{1}{2}|Du^2|dS\end{aligned}$$

Now (recall that $\frac{\partial u}{\partial\nu} := \nu\cdot Du$ and $|\nu| = 1$)

$$\left|\frac{\partial u}{\partial\nu}\cdot u_t\right| \le |u_t|\cdot|Du| \le \frac{1}{2}u_t^2 + \frac{1}{2}|Du|^2$$

by the Cauchy-Schwarz Inequality and Cauchy's Inequality

Now we place this inequality into our last result for $\dot{e}(t)$ to obtain

$$\dot{e}(t) = \int_{\partial B(x_0,t_0-t)} u_t\frac{\partial u}{\partial\nu}dS - \left(\frac{1}{2}(u_t^2) + \frac{1}{2}|Du^2|dS\right) \le \int_{\partial B(x_0,t_0-t)} 0$$

So $\dot{e}(t) \le 0$; and so $e(t) \le e(0) = 0$ (since $t = 0$ gives us the region where we assumed $u_t, u \equiv 0$) for all $0 \le t \le t_0$. Thus $u_t, Du \equiv 0$ (by the definition of $e(t)$).

Lastly, $u_t \equiv 0$ implies that $u(x,t)$ is constant in time, and $Du \equiv 0$implies $u(x,t)$ is constant in space for all $0 \le t \le t_0$. Thus $u(x,t) = C$ for some constant $C$. Then, from the assumption $u \equiv 0$ on $B(x_0,t_0) \times \{t = 0\}$, so in order for $u(x,t)$ to be constant, it must be equal to $0$.

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