Fall 2025 Solutions

Problem 1

Let u be harmonic on $\mathbb{R}^n$ and satisfies the following: There exists a constant $C > 0$ such that:

$$\int_{\{y\in\mathbb{R}^n:|y-x|<1\}} |u(y)|dy \leq C$$

for each $x \in \mathbb{R}^n$. Prove that u is constant

proof:
Since u is harmonic we can apply the

> Mean Value Formula
>
> **Evans p.25:**
> If $u \in C^2(U)$ is harmonic, then
>
> $$u(x) = ⨍_{\partial B(x,r)} u(y)dS(y) = ⨍_{B(x,r)} u\, dy$$
>
> for each ball $B(x,r) \subset U$
>
> #PDE

which tells us

$$|u(x)| = |⨍_{B(x,1)} u(y)\, dy| \leq ⨍_{B(x,1)} |u(y)|\, dy = \frac{1}{\alpha(n)} \int_{B(x,1)} |u(y)|\, dy \leq \frac{C}{\alpha(n)}$$

for all $x \in \mathbb{R}^n$ and for all $B(x,1) \subset \mathbb{R}^n$. Thus, since $u(x)$ is both harmonic, and bounded on $\mathbb{R}^n$, by

> Liouville's Theorem
>
> **Statement:**
> A function which is analytic and bounded in the whole plane must reduce to a constant.
>
> #Complex_Analysis

it is a constant function.
■

Problem 2

Consider the following Cauchy Problem:

$$\begin{aligned} u_x + u_y = u^2 \text{ in } \{(x,y) \in \mathbb{R}^2 : y > -x, x > 0\} \\ u(x,-x) = x, \quad x > 0 \end{aligned}$$

(a) Use the method of characteristics to find an explicit formula for u
(b) Show that the solution becomes infinite along the hyperbola $x^2 - y^2 = 4$

(a)

proof:
Using the method of characteristics we have

$$\frac{dx}{ds} = 1, \quad \frac{dt}{ds} = 1, \quad \frac{dz}{ds} = z^2$$

Thus

$$\begin{aligned} x = s + c_1(r), \quad y = s + c_2(r) \\ \implies x(r,-r) = -r + c_1(r) = r, \quad y(r,-r) = -r + c_1(r) = -r \\ \implies x(r,s) = s + 2r, \qquad\qquad y(r,s) = s \end{aligned}$$

Therefore $x - y = 2r \implies r(x,y) = \frac{1}{2}(x-y)$ and $s(x,y) = y$. Solving for z we have

$$-\frac{1}{z} = s + c_3(r) \implies z(r,s) = \frac{-1}{c_3(r) + s}$$

Using initial data we have

$$\begin{aligned} z(r,-r) = \frac{-1}{c_3(r) + (-r)} = r \\ \implies c_3(r) = \frac{r^2-1}{r} \\ \implies z(r,s) = \frac{-1}{\frac{r^2-1}{r} + s} = \frac{-r}{r^2 - 1 + rs} \end{aligned}$$

Now we may plug in our values of $r(x,y)$ and $s(x,y)$ to obtain

$$\begin{aligned} z(r,s) = z(r(x,y), s(x,y)) = u(x,y) = \frac{-\frac{1}{2}(x-y)}{\left(\frac{(x-y)}{2}\right)^2 - 1 + \frac{x-y}{2}y} \\ = \frac{-(x-y)}{\frac{x^2}{2} + \frac{y^2}{2} - xy + xy - y^2 - 2} \\ = \frac{-(x-y)}{\frac{1}{2}(x^2-y^2) - 2} \\ = \frac{2(x-y)}{4 - (x^2 - y^2)} \end{aligned}$$

(b)

Show that the solution becomes infinite along the hyperbola $x^2 - y^2 = 4$

<u>proof:</u>
Along the characteristic

$$z(r,s) = \frac{1}{\frac{1-r^2}{r} - s}$$

the solution will blow up when $\frac{1-r^2}{r} = s$ and thus when $1 - r^2 = rs$. Now plugging in x and y we have

$$\begin{aligned} 1 - \frac{(x-y)^2}{4} = \frac{(x-y)y}{2} \\ \implies 4 - (x^2 - 2xy + y^2) = 2xy - 2y^2 \\ \implies 4 = x^2 - y^2 \end{aligned}$$

Thus, the solution becomes infinite along the hyperbola $x^2 - y^2 = 4$.
■

Problem 3

Let $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ be the periodic box with $|\mathbb{T}^3| = 1$ and let $v = v(x)$ be a given divergence-free (i.e. $\nabla \cdot v = 0$) periodic, smooth vector field. Assume that $\theta(t, x)$ is a periodic, smooth function solving

$$\theta_t + v \cdot \nabla\theta = \Delta\theta, \quad x \in \mathbb{T}^3,\ t > 0$$
$$\theta(0, x) = \theta_0(x)$$

(a). Show that

$$\frac{d}{dt}\int_{\mathbb{T}^3} \theta dx = 0$$

(b). Denote the average $\frac{1}{\mathbb{T}^3}\int_{\mathbb{T}^3} \theta\, dx$ by $\bar{\theta}$. Prove that there exists a constant $c > 0$ such that

$$\|\theta(t, \cdot) - \bar{\theta}\|_{L^2} \le e^{-ct}\|\theta_0(\cdot) - \bar{\theta}\|_{L^2} \quad \text{for all } t > 0$$

Hint: Compute $\frac{d}{dt}\int_{\mathbb{T}^3} |\theta(t, x) - (\bar{\theta})|^2\, dx$.

(a)

proof:
Take

$$\frac{d}{dt}\int_{\mathbb{T}^3} \theta\, dx = \int_{\mathbb{T}^3} \frac{d}{dt}\theta\, dx = \int_{\mathbb{T}^3} \theta_t\, dx = \int_{\mathbb{T}^3} \Delta\theta - v \cdot \nabla\theta\, dx$$

By

> Green's Formulas
>
> Let $u, v \in C^2(\bar{U})$
> (i)
>
> $$\int_U \Delta u\, dx = \int_{\partial U} \frac{\partial u}{\partial \nu} dS$$
>
> (ii)
>
> $$\int_U \nabla v \cdot \nabla u\, dx = -\int_U u\Delta v\, dx + \int_{\partial U} u\frac{\partial v}{\partial \nu} dS$$
>
> (iii)
>
> $$\int_U u\Delta v - v\Delta u\, dx = \int_{\partial U} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} dS$$
>
> #PDE

we have

$$\int_{\mathbb{T}^3} \Delta\theta\, dx = \int_{\partial\mathbb{T}^3} \frac{\partial\theta}{\partial\nu}\, dS = 0$$

since the torus has no boundary. Thus

$$\frac{d}{dt}\int_{\mathbb{T}^3} \theta\, dx = -\int_{\mathbb{T}^3} v \cdot \nabla\theta\, dx$$

Note that

$$\begin{aligned} \operatorname{div}(\theta v) = \nabla \cdot (\theta v) = \sum_{i=1}^{n} \partial_i(\theta v^i) = \sum_{i=1}^{n}((v^i\partial_i\theta) + (\theta\partial_i v^i)) = (v \cdot \nabla\theta) + (\theta(\nabla \cdot v)) \\ = v \cdot \nabla\theta + \theta\operatorname{div}(v) \end{aligned}$$

Therefore, since v is divergence free, we have that $v \cdot \nabla\theta = \operatorname{div}(\theta v) = \nabla \cdot (\theta v)$. As a result

$$\frac{d}{dt}\int_{\mathbb{T}^3}\theta\,dx = -\int_{\mathbb{T}^3} v\cdot\nabla\theta\,dx = -\int_{\mathbb{T}^3}\operatorname{div}(\theta v)$$

Then by

> Gauss-Green
>
> **Evans p.711**
>
> (i) Suppose $u \in C^1(\overline{U})$. Then
>
> $$\int_U u_{x_i}\,dx = \int_{\partial U} u\nu^i\,dS \quad (i = 1,\dots,n).$$
>
> (ii) - also called the divergence theorem
>
> $$\int_U \operatorname{div} u\,dx = \int_{\partial U} u\cdot\nu\,dS$$
>
> for each vector field $u \in C^1(\overline{U};\mathbb{R}^n)$.
>
> #PDE

we have

$$\frac{d}{dt}\int_{\mathbb{T}^3}\theta\,dx = -\int_{\mathbb{T}^3}\operatorname{div}(\theta v)\,dx = -\int_{\partial\mathbb{T}^3}\theta v\cdot\nu\,dS = 0$$

since the torus has no boundary.
■

(b)

proof:
Note that $\bar{\theta}$ is purely a function of time. Set

$$\phi(t,x) := \theta(t,x) - \bar{\theta}$$

From part (a), $\bar{\theta}$ is constant in time, so

$$\phi_t = \theta_t = \Delta\theta - v\cdot\nabla\theta$$

Then since $\bar{\theta}$ is constant in space,

$$\begin{aligned}\Delta(\phi) - v\cdot\nabla\phi &= \Delta(\theta-\bar{\theta}) - v\cdot\nabla(\theta-\bar{\theta})\\ &= \Delta(\theta) - \Delta(\bar{\theta}) - v\cdot\nabla(\theta) - v\cdot\nabla(\bar{\theta})\\ &= \Delta(\theta) - v\cdot\nabla(\theta)\end{aligned}$$

As a result $\phi_t = \Delta(\phi) - v\nabla(\phi)$. Now we may evaluate

$$\frac{d}{dt}\|\phi(t)\|^2_{L^2} = \frac{d}{dt}\int_{\mathbb{T}^3}|\phi(t,x)|^2dx$$

as we would in an energy method

$$\begin{aligned}\left(\frac{1}{2}\right)\frac{d}{dt}\int_{\mathbb{T}^3}|\phi(t,x)|^2\,dx &= \int_{\mathbb{T}^3}\phi\phi_t\,dx\\ &= \int_{\mathbb{T}^3}\phi(\Delta\phi - v\cdot\nabla\phi)\,dx\\ &= \int_{\mathbb{T}^3}\phi\Delta\phi\,dx - \int_{\mathbb{T}^3}\phi(v\cdot\nabla\phi)\,dx\\ &= -\int_{\mathbb{T}^3}\nabla|\phi|^2 - \int_{\mathbb{T}^3}\phi(v\cdot\nabla\phi\,)dx\end{aligned}$$

From

> Green's Formulas
>
> Let $u, v \in C^2(\bar{U})$
> (i)

$$\int_U \Delta u \, dx = \int_{\partial U} \frac{\partial u}{\partial \nu} dS$$

(ii)

$$\int_U \nabla v \cdot \nabla u \, dx = -\int_U u \Delta v \, dx + \int_{\partial U} u \frac{\partial v}{\partial \nu} dS$$

(iii)

$$\int_U u \Delta v - v \Delta u \, dx = \int_{\partial U} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} dS$$

#PDE

and the torus having no boundary.

On the right side of the difference we have $\phi(v \cdot \nabla \phi) = (\phi v) \cdot \nabla \phi$. As we have proven in part (a) $v \cdot \nabla(\phi) = \nabla(\phi v) - \phi \text{div}(v) = \nabla(\phi v)$. Thus

$$\begin{aligned}(\phi v) \cdot \nabla \phi = \nabla(\phi^2 v) - \phi \nabla(v\phi) \\ = \nabla(\phi^2 v) - \phi(v \cdot \nabla \phi) + \phi^2(\text{div}(v)) \\ = \nabla(\phi^2 v) - ((\phi v) \cdot \nabla \phi) \\ \Rightarrow (\phi v) \cdot \nabla \phi = \frac{1}{2} \nabla(\phi^2 v)\end{aligned}$$

Then

$$\int_{\mathbb{T}^3} \phi(v \cdot \nabla \phi) = \left(\frac{1}{2}\right) \int_{\mathbb{T}^3} \nabla \cdot (\phi^2 v) \, dx = \frac{1}{2} \int_{\mathbb{T}^3} \text{div}(\phi^2 v) = \int_{\partial \mathbb{T}^3} \phi^2 v \cdot \nu dS = 0$$

since the torus has no boundary. As a result

$$\frac{d}{dt} \|\phi(t)\|_{L^2}^2 = -2 \int_{\mathbb{T}^3} |\nabla \phi|^2 = -2 \|\nabla \phi(t)\|_{L^2}^2 \leq -c \|\phi(t)\|_{L^2}^2$$

with the last inequality following from the

> Poincare Inequality

Evans p.290

(i)
Let U be a bounded, connected, open subset of $\mathbb{R}^n$ with a C^1 boundary ∂U. Assume $1 \leq p \leq \infty$. Then there exists a constant C depending only on n, p and U such that

$$\|u - ⨍_U u \, dy\|_{L^p(U)} \leq C \|Du\|_{L^p(U)}$$

(ii)
This is also sometimes called *Poincare's Inequality*

Assume U is a bounded, open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(U)$ for some $1 \leq p < n$. Then we have the estimate

$$\|u\|_{L^q(U)} \leq C \|Du\|_{L^p(U)}$$

for each constant $q \in [1, p^*]$, (where $p^* = \frac{np}{n-p}$) the constant C depending only on p, q, n and U.

In particular, for all $1 \leq p \leq \infty$

$$\|u\|_{L^p(U)} \leq C \|Du\|_{L^P(U)}$$

#PDE

. As a result we have

$$\frac{d}{dt} \|\phi(t)\|_{L^2}^2 \leq -c \|\phi(t)\|_{L^2}^2$$

so by

> Gronwall's Inequality

Simplified Versions From Evans

Evans p. 708

Differential Form

(i) Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0, T]$ which satisfies a.e. t the differential inequality

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t)$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions (integrable) on $[0, T]$. Then

$$\eta(t) \leq e^{\int_0^t \phi(s) ds} \left[\eta(0) + \int_0^t \psi(s)\, ds\right]$$

for all $0 \leq t \leq T$.

(ii) In particular, if

$$\eta' \leq \phi\eta \text{ on [0,T]} \quad \text{and} \quad \eta(0) = 0,$$

then

$$\eta \equiv 0 \text{ on } [0, T]$$

Integral Form

(i) Let $\xi(t)$ be a nonnegative, summable function on $[0, T]$ which satisfies for a.e. t the integral inequality

$$\xi(t) \leq C_1 \int_0^t \xi(s)\, ds + C_2$$

for constants $C_1, C_2 \geq 0$. Then

$$\xi(t) \leq C_2(1 + C_1 t e^{C_1 t})$$

for a.e. $0 \leq t \leq T$

(ii) In particular, if

$$\xi(t) \leq C_1 \int_0^t \xi(s)\, ds$$

for a.e. $0 \leq t \leq T$, then

$$\xi(t) = 0 \text{ a.e}$$

Sign Agnostic Version

Differential Form:
Let I denote an interval of the real line of the form $[a, b), [a, b]$, or $[a, \infty)$ and let β, u be real-valued continuous functions defined on I. If u is differentiable in the interior I^o of I and satisfies

$$u'(t) \leq \beta(t)u(t) \text{ for } t \in I^o$$

then u is bounded by the solution of the corresponding differential equation $v'(t) = \beta(t)v(t)$:

$$u(t) \leq u(a)e^{\int_a^t \beta(s) ds}$$

for all $t \in I$.

Remark: There are no assumptions on the signs of the functions $\beta$and u.

<u>proof:</u>
Let $u'(t) \leq \beta(t)u(t)$. Now define

$$v(t) = e^{\int_0^t \beta(s)\, ds} \Rightarrow v'(t) = \beta(t)v(t)$$

Then by the quotient rule

$$\frac{d}{dt}\frac{u}{v} = \frac{v(t)(u'(t) - \beta(t)u(t))}{v^2(t)} \leq 0$$

Therefore

$$\frac{u(t)}{v(t)} \leq \frac{u(0)}{v(0)} \Rightarrow u(t) \leq u(0)e^{\int_a^t \beta(s)\, ds}$$

■

#PDE

$$\|\phi(t)\|_{L^2}^2 \leq \|\phi(0)\|_{L^2}^2 e^{\int_0^t -c\, ds} = \|\theta_0 - \bar{\theta}\|_{L^2}^2 e^{-ct}$$

Thus

$$\|\theta - \bar{\theta}\|_{L^2} \leq e^{-ct}\|\theta_0 - \bar{\theta}\|_{L^2}$$

■