PDE Screening Exam
Fall 2025

Partial credit will be awarded, but in the event that you can not fully solve a problem you should state clearly what it is you have done and what you have left out. Start each problem on a fresh sheet of paper and write on only one side of the paper.

1. Suppose u is harmonic on $\mathbb{R}^n$, and satisfies the following property: there exists a constant $C > 0$ such that

$$\int_{\{y\in\mathbb{R}^n:|y-x|<1\}} |u(y)|dy \leq C \text{ for each } x \in \mathbb{R}^n.$$

   Prove that u is constant.

2. Consider the following Cauchy problem:

$$\partial_x u + \partial_y u = u^2 \text{ in } \{(x,y)\in\mathbb{R}^2 : y > -x,\ x > 0\},$$
$$u(x,-x) = x,\ x > 0.$$

   (a) Use the method of characteristics to find an explicit formula for u.

   (b) Show that the solution becomes infinite along the hyperbola $x^2 - y^2 = 4$.

3. Let $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ be the periodic box with $|\mathbb{T}^3| = 1$, and let $v = v(x)$ be a given divergence-free (i.e. $\nabla \cdot v = 0$), periodic, smooth vector field. Assume that $\theta(t,x)$ is a periodic, smooth function solving

$$\partial_t\theta + v\cdot\nabla\theta = \Delta\theta, \quad x\in\mathbb{T}^3,\ t>0,$$
$$\theta(0,x) = \theta_0(x).$$

   (a) Show that

$$\frac{d}{dt}\int_{\mathbb{T}^3}\theta dx = 0.$$

   (b) Denote the average $\frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3}\theta dx = \int_{\mathbb{T}^3}\theta dx$ by $\overline{\theta}$. Prove that there exists a constant $c > 0$ such that

$$\|\theta(t,\cdot)-\overline{\theta}\|_{L^2} \leq e^{-ct}\|\theta_0(\cdot)-\overline{\theta}\|_{L^2} \text{ for all } t>0.$$

   Hint: Compute $\frac{d}{dt}\int_{\mathbb{T}^3}|\theta(t,x)-\overline{\theta}|^2dx$.