Fall 2024 Solutions

Problem 1

Consider a linear transport equation ($a > 0$) with drag ($b > 0$):

$$u_t + au_x + bu = 0$$

(a) Find the solution $u(t,x)$ to Eq. (1) for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ with the initial condition $u(0,x) = f(x)$
(b) Find the solution $u(t,x)$ to Eq. (1) for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}_+$ subject to the initial condition $u(0,x) = f(x)$ and the boundary condition $u(t,0) = g(t)$, where $f(0) = g(0)$.

(a)

proof:

$F(x,t,z,p,q) = q + ap + bz = 0$
Thus our characteristics ODEs are:

$$\frac{\partial x}{\partial s} = F_p = a, \quad \frac{\partial t}{\partial s} = F_q = b, \quad \frac{\partial z}{\partial s} = pF_p + qF_q = -bz$$

With initial conditions:

$$x(0,r) = r, \quad t(0,r), \quad z(0,r) = f(r)$$

Then:

$$x(s,r) = as + r, \quad t(s,r) = s, \quad z(r,s) = f(r)e^{-bs}$$

Thus:

$$r(t,x) = x - at, \quad s(t,x) = t, \quad u(t,x) = z(r(t,x), s(t,x)) = z(x - at, t) = f(x - at)e^{-bt}$$

So as a result:
$u(t,x) = f(x - at)e^{-bt}$ ■

(b)

proof:
Find the solution $u(t,x)$ to Eq. (1) for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}_+$ subject to the initial condition $u(0,x) = f(x)$ and the boundary condition $u(t,0) = g(t)$, where $f(0) = g(0)$.

Solution:

The linear transport equation $u_t + au_x + bu = 0$ propagates information along characteristic curves defined by

$$\frac{dx}{dt} = a$$

These are straight lines in the (t,x) plane with slope a

$$x = at + r$$

Where r is a constant that determines where it originates:

1) If $r \geq 0$, then the characteristic starts from $t = 0$ (i.e. $x = r$ at $t = 0$)

2) If $r < 0$ then the characteristic starts from $x < 0$, but since we are restricted to $x \geq 0$, it instead starts from the boundary condition at $x = 0$ (i.e. $x = 0, t = -\frac{r}{a} = \tau$)

So for 1. we have already solved, and for 2. we have

$$t(\tau, 0) = \tau \quad x(\tau, 0) = 0 \quad z(0, \tau) = g(\tau)$$

Thus $t(s, \tau) = s + c_2(\tau) \Rightarrow t(s, 0) = \tau$ and thus $t = s + \tau$. Then $x(s, \tau) = as + c_1(\tau) \Rightarrow x(s, 0) = as$. Lastly, $z(s, \tau) = g(\tau)e^{-2s}$ As a result

$$s = \frac{x}{a}, \quad \tau = t - \frac{x}{a}$$

and we have

$$z(s(x,t), \tau(x,t)) = z\left(\frac{x}{a}, t - \frac{x}{a}\right) = g\left(t - \frac{x}{a}\right)e^{-bt}$$

Thus our final result is:

$u(t,x) = \begin{cases} f(x - at)e^{-bt}, & x \geq at \\ g\left(t - \frac{x}{a}\right)e^{-bt}, & x < at \end{cases}$

■

Problem 2

Let $\Omega \subset \mathbb{R}^n$ with $n > 1$ be a bounded domain with smooth boundary. Assume $u \in C^2(\bar{\Omega})$ solves

$$\Delta u = u^7 + 2u^5 + 3u \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

Show that u is identically 0

proof:

Method 1: Green's First Identity

By

> Green's Formulas
>
> Let $u, v \in C^2(\bar{U})$
>
> (i)
>
> $$\int_U \Delta u \, dx = \int_{\partial U} \frac{\partial u}{\partial \nu} dS$$
>
> (ii)
>
> $$\int_U \nabla v \cdot \nabla u \, dx = -\int_U u \Delta v \, dx + \int_{\partial U} u \frac{\partial v}{\partial \nu} dS$$
>
> (iii)
>
> $$\int_U u\Delta v - v\Delta u \, dx = \int_{\partial U} u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu} dS$$
>
> #PDE

we have that for $u, v \in C^2(\bar{U})$

$$\int_U \nabla v \cdot \nabla u \, dx = -\int_U u\Delta v \, dx + \int_{\partial U} u\frac{\partial u}{\partial \nu} dS$$

Thus, setting $v = u$ we have

$$\int_U |\nabla u|^2 \, dx = -\int_U u\Delta u \, dx + \int_{\partial U} u\frac{\partial u}{\partial \nu} dS$$

Note that since $u \equiv 0$ on $\partial\Omega$ then

$$\int_{\partial U} u\frac{\partial u}{\partial \nu} dS = 0$$

Furthermore

$$\Delta u = u^7 + 2u^5 + 3u \Rightarrow u\Delta u = u^8 + 2u^6 + 3u^2$$

we have

$$\int_U u\Delta u \, dx = \int_U u^8 + 2u^6 + 3u^2 \, dx$$

Thus

$$\int_U |\nabla u|^2 \, dx = -\int_U u^8 + 2u^6 + 3u^2 \, dx$$

The left hand side must be positive, and the right hand side (as the sum of even powers of real numbers multiplied by a negative) must be negative. The only way this is possible is if

$$\int_U |\nabla u|^2 \, dx = -\int_U u^8 + 2u^6 + 3u^2 \, dx = 0$$

Therefore $|\nabla u|^2 = u^8 + 2u^6 + 3u^2 = 0$ a.e. which implies that $u = 0$ a.e.

Since u is continuous and equal to 0 a.e., then by continuity $u \equiv 0$ everywhere in Ω ■

Method 2: Elliptic PDE Max Principle

Since Δ is already a uniformly elliptic operator and Ω is a domain, we can apply the strong maximum principle directly. Call $\Delta = L$ here for our uniformly elliptic operator. We have two cases where u could potentially be non-zero on Ω, since $u < 0 \Rightarrow Lu < 0$ and $u > 0 \Rightarrow Lu > 0$

Case 1 (u < 0)

Assume $u < 0$ somewhere on Ω, then $Lu = u^7 + 2u^5 + u^3 < 0$, and u then attains its minimum over $\bar{\Omega}$ at an interior point ($u = 0$ on the boundary), then by the strong maximum principle u is constant within Ω, by the continuity of u this is a contradiction. Thus $u \equiv 0$.

Case 2 (u > 0)

Assume $u > 0$ somewhere on Ω, then $Lu = u^7 + 2u^5 + u^3 > 0$ and u then attains its maximum over $\bar{\Omega}$ at an interior point (since $u = 0$), then by the strong maximum principle u is constant within Ω. By the continuity of u, this is a contradiction. Thus $u \equiv 0$.

Problem 3

Let $\Omega \subset \mathbb{R}^n$ with $n > 1$ be a bounded domain with smooth boundary. Let $u \in C^2([0,\infty) \times \bar{\Omega})$ which solves the equation

$$u_{tt} - \Delta u = u$$

with the boundary condition $u = 0$ on $\partial\Omega$. Let

$$E(t) = \frac{1}{2}\int_\Omega u_t^2(t,x) + |\nabla_x u(t,x)|^2 dx$$

Prove that there exists $C > 0$ independent of t such that

$$E(t) \le \exp(Ct)E(0) \quad \text{for } t \ge 0$$

proof:
Note that $E(t)$ is the "energy" of the wave equation.

Thus

$$\frac{d}{dt}E(t) = \int_\Omega u_t(u_{tt} - \Delta u)\, dx$$

Then, since $u_{tt} - \Delta u = u$ we have that

$$\frac{d}{dt}E(t) = \int_\Omega u_t u\, dx$$

Then we use

> Holder's Inequality
>
> Let $1 \le p, q \le \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then if $f \in L^p(U)$ and $g \in L^q(U)$
>
> $$\int_U |fg|\, dx \le \|f\|_{L^p(U)} \cdot \|g\|_{L^q(U)}$$

to say

$$2\frac{d}{dt}E(t) = \left|\int_\Omega u_t u\, dx\right| \le \int_\Omega |u_t u|\, dx \le \|u_t\|_{L^2(\Omega)} \cdot \|u\|_{L^2(\Omega)}$$

Observe that

$$\|u_t\|_{L^2(\Omega)}^2 = \left(\left(\int_\Omega |u_t|^2\right)^{1/2}\right)^2$$
$$= \int_\Omega |u_t|^2 \le \int_\Omega |u_t|^2 + |\nabla_x u(t,x)|^2\, dx = 2E(t)$$

Thus

$$\|u_t\|_{L^2(\Omega)} \le \sqrt{2E(t)}$$

Then we have from the

> Poincare Inequality
>
> **Evans p.290**
>
> (i)
> Let U be a bounded, connected, open subset of $\mathbb{R}^n$ with a C^1 boundary ∂U. Assume $1 \le p \le \infty$. Then there exists a constant C depending only on n, p and U such that
>
> $$\|u - ⨍_U u\, dy\|_{L^p(U)} \le C\|Du\|_{L^p(U)}$$

(ii)
This is also sometimes called *Poincare's Inequality*

Assume U is a bounded, open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(U)$ for some $1 \leq p < n$. Then we have the estimate

$$\|u\|_{L^q(U)} \leq C\|Du\|_{L^p(U)}$$

for each constant $q \in [1, p^*]$, (where $p^* = \frac{np}{n-p}$) the constant C depending only on p, q, n and U.

In particular, for all $1 \leq p \leq \infty$

$$\|u\|_{L^p(U)} \leq C\|Du\|_{L^p(U)}$$

#PDE

that there exists some C (not dependent on t) such that

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}$$

Once again comparing this to $E(t)$ we have

$$\begin{aligned}\|u\|^2_{L^2(\Omega)} \leq C^2\|\nabla u\|^2_{L^2(\Omega)} = C^2\int_\Omega |\nabla u|^2\,dx \leq \int_\Omega |u_t|^2 + |\nabla_x u(t,x)|^2\,dx \\ = 2E(t) = 2C^2E(t)\end{aligned}$$

As a result

$$\|u\|_{L^2(\Omega)} \leq C\sqrt{2E(t)}$$

Putting this all together we have

$$\frac{d}{dt}E(t) \leq \|u_t\|_{L^2(\Omega)} \cdot \|u\|_{L^2(\Omega)} \leq \sqrt{2E(t)} \cdot C\sqrt{2E(t)} = 2CE(t)$$

Let $C' = 2C$ Lastly, we utilize the derivative version of

Gronwall's Inequality

Simplified Versions From Evans

Evans p. 708

Differential Form

(i) Let $\eta(\cdot)$ be a nonnegative, absolutely continuous function on $[0, T]$ which satisfies a.e. t the differential inequality

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t)$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions (integrable) on $[0, T]$. Then

$$\eta(t) \leq e^{\int_0^t \phi(s)ds}\left[\eta(0) + \int_0^t \psi(s)\,ds\right]$$

for all $0 \leq t \leq T$.

(ii) In particular, if

$$\eta' \leq \phi\eta \text{ on [0,T]} \quad \text{and} \quad \eta(0) = 0,$$

then

$$\eta \equiv 0 \text{ on } [0, T]$$

Integral Form

(i) Let $\xi(t)$ be a nonnegative, summable function on $[0,T]$ which satisfies for a.e. t the integral inequality

$$\xi(t) \leq C_1 \int_0^t \xi(s)\, ds + C_2$$

for constants $C_1, C_2 \geq 0$. Then

$$\xi(t) \leq C_2(1 + C_1 t e^{C_1 t})$$

for a.e. $0 \leq t \leq T$

(ii) In particular, if

$$\xi(t) \leq C_1 \int_0^t \xi(s)\, ds$$

for a.e. $0 \leq t \leq T$, then

$$\xi(t) = 0 \text{ a.e}$$

Sign Agnostic Version

Differential Form:
Let I denote an interval of the real line of the form $[a,b), [a,b],$ or $[a,\infty)$ and let β, u be real-valued continuous functions defined on I. If u is differentiable in the interior I^o of I and satisfies

$$u'(t) \leq \beta(t)u(t) \text{ for } t \in I^o$$

then u is bounded by the solution of the corresponding differential equation $v'(t) = \beta(t)v(t)$:

$$u(t) \leq u(a)e^{\int_a^t \beta(s)ds}$$

for all $t \in I$.

Remark: There are no assumptions on the signs of the functions $\beta$and u.

proof:
Let $u'(t) \leq \beta(t)u(t)$. Now define

$$v(t) = e^{\int_0^t \beta(s)\, ds} \Rightarrow v'(t) = \beta(t)v(t)$$

Then by the quotient rule

$$\frac{d}{dt}\frac{u}{v} = \frac{v(t)(u'(t) - \beta(t)u(t))}{v^2(t)} \leq 0$$

Therefore

$$\frac{u(t)}{v(t)} \leq \frac{u(0)}{v(0)} \Rightarrow u(t) \leq u(0)e^{\int_a^t \beta(s)\, ds}$$

■

#PDE

to state that since

$$\frac{dE}{dt} \leq C'E(t) + 0$$

Then

$$E(t) \leq e^{\int_0^t C'\, ds}[E(0) + 0] = e^{C't}E(0)$$

for all $t \geq 0$. We have proven the result ■