Geometry/Topology qualifying exam
Spring 1999

1. Let M be an embedded compact surface in $\mathbb{R}^3$, namely a non-empty 2–dimensional submanifold of $\mathbb{R}^3$. Show that there exists an infinite number of vertical lines $L = \{x\} \times \{y\} \times \mathbb{R}$ which meet M and are not tangent to it in the following sense: $M \cap L$ is non-empty and, for every $x \in M \cap L$, the plane tangent to M at x is not vertical.

2. Let N be a n–dimensional submanifold of the m–dimensional manifold M, and let $i : N \to M$ be the inclusion map. Suppose that N is closed in M. Show that, if $\alpha \in \Omega^p(N)$ is a degree p differential form on N, there exists a form $\beta \in \Omega^p(M)$ on M such that $i^*(\beta) = \alpha$. If $d\alpha = 0$, can you always choose β so that $d\beta = 0$?

3. In $B^2 \times B^2$, let X be the union of the torus $S^1 \times S^1$ and of the disk $B^2 \times \{x\}$ (where B^2 is the closed unit disk in $\mathbb{R}^2$ and S^1 is its boundary circle). Compute the fundamental group of X.

4. For X as in Problem 3, compute the homology modules $H_p(X; R)$, with coefficients in an arbitrary unitary ring R.

5. Prove or disprove: A surjective map $p : \widetilde{X} \to X$ is a covering map if and only if, for every $\widetilde{x} \in \widetilde{X}$, there is a neighborhood $\widetilde{U}$ of $\widetilde{x}$ such that the restriction $p_{|\widetilde{U}} : \widetilde{U} \to p\left(\widetilde{U}\right)$ is a homeomorphism.

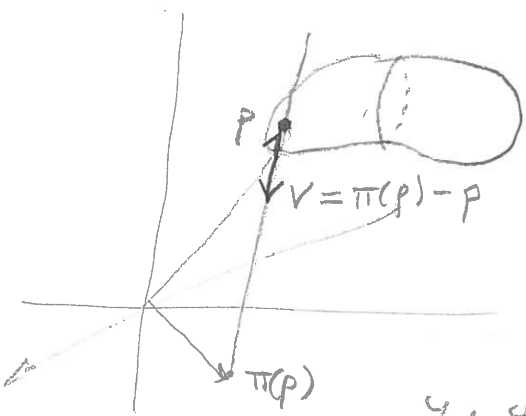
6. Let X be a path connected space such that $\pi_1(X; x_0) = 1$ and $\pi_2(X; x_0) = 1$. Recall that the second property means that, for every continuous map $\alpha : [0,1] \times [0,1] \to X$ such that $\alpha(s,t) = x_0$ when $s \in \{0,1\}$ or $t \in \{0,1\}$, there is a homotopy $H : [0,1] \times [0,1] \times [0,1] \to X$ such that $H(s,t,u) = x_0$ when $s \in \{0,1\}$ or $t \in \{0,1\}$ or $u = 1$. Consider the 2–dimensional torus $T^2 = S^1 \times S^1$. Show that every continuous map $f : T^2 \to X$ is homotopic to a constant map. (Possible hint: Write the torus as a square with identifications of its sides.)

1. Use Sard's Theorem on $\pi : M \to \{xy\text{-plane}\}$
2. Use a bump function to extend α smoothly
3. Van Kampen
4. Cellular Homology
5. False: Consider countably many copies of $\mathbb{R}$ covering a shrinking wedge of circles.
6. f is homotopic to a map sending edges of to x_0 by $\pi_1(X, x_0) = 1$.

Then use $\pi_2(X, x_0) = 1$.

1. $M \subseteq \mathbb{R}^3$ emb, cpt, 2-mfld

Show $\{L = \{x\}\times\{y\}\times\mathbb{R} : L$ not tangent to $M\} = \infty$



$\pi : \mathbb{R}^3 \to \mathbb{R}^2 \subseteq \mathbb{R}^3$

$(x,y,z) \mapsto (x,y,0)$

$T_p\pi : T_pM \cong \mathbb{R}^2 \longrightarrow T_p\mathbb{R}^2 \cong \mathbb{R}^2$

π linear $\Rightarrow T_p\pi = \pi \Rightarrow T_p\pi(\pi(p)-p) = \pi(p)-\pi(p)$
$= 0$

Let $\mathcal{L}$ be our family & consider the complement:

$\mathcal{L}^c = \{L : L$ tangent to $M\} = \{V : V = \pi(p)-p \in T_pM$ for some $p\}$
$\subseteq \{V : V \in \ker T_p\pi$ for some $p\}$

For a map, $\ker T_p\pi \neq 0 \iff p$ critical point of π, so, $= \coprod_{\substack{p\in M \\ p \text{ crit of } \pi}} \ker T_p\pi$

$0 = \mu(\{p\in M : p \text{ critical}\}) = \mu(\{p \in M : \ker T_p\pi \neq 0\})$
Sard

indexed by meas. 0 set, hence meas. 0
?

(2.) $N \subseteq M$, $\dim N = n$, $\dim M = m$, $i: N \to M$ inclusion

Suppose N closed in M.

Show that, if $\alpha \in \Omega^p(N)$, $\exists \beta \in \Omega^p(M)$ s.t. $i^*(\beta) = \alpha$. If $d\alpha = 0$, can you always choose β s.t. $d\beta = 0$?

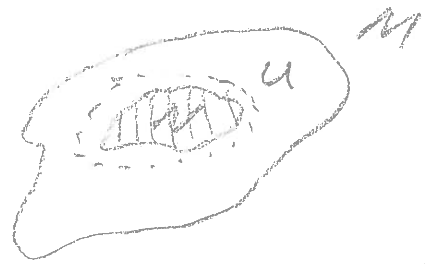
WTS: $i^*: \Omega^p(M) \longrightarrow \underset{\alpha}{\Omega^p(N)}$ surjective.

#1

$\alpha_p \in Alt^p(T_pN) \Rightarrow \alpha_p : T_pN \otimes \cdots \otimes T_pN \longrightarrow \mathbb{R}$

$\beta_p : T_pM \otimes \cdots \otimes T_pM \to \mathbb{R}$

$i^*(\beta) = \beta_{i(p)}(T_p i\, v)$



$N \subseteq M$ closed.

$\beta_p(v) = \alpha_p(v)$ for $v \in \underset{\cong \mathbb{R}^n}{T_pN} \subseteq \underset{\cong \mathbb{R}^m}{T_pM}$ subspace

$=$ 

$= 0$ for $v \in (T_pN)^\perp$

Then $i^*(\beta) = \alpha$

#2: i^* surj on forms $\Rightarrow$ i^* surj on homology

$\Rightarrow$ $d\alpha = 0 \Rightarrow \alpha \in H^p(N) \Rightarrow \exists \beta \in H^p(M)$ s.t. $i^*(\beta) = \alpha$

But $0 = d\alpha = d\, i^*(\beta) = i^*(d\beta) =$

$\Rightarrow d\beta = 0$ since hom.

③ $X = (S^1 \times S^1 \amalg B^2) / S^1 \times \{x\} \sim \partial B^2$

$\pi_1(X) = ?$

Let $A = T^2$, $B = B^2$, $A \cap B = S^1$, $A \cup B = X$

$i_1 : A \cap B \hookrightarrow A$

$i_1^* : \pi_1(A \cap B) \longrightarrow \pi_1(A) \cong \mathbb{Z}\langle a \rangle \oplus \mathbb{Z}\langle b \rangle$

$i_1^*(\alpha) = a$

$i_2^* : \pi_1(A \cap B) \longrightarrow \pi_1(B) = 1 \quad \Rightarrow \quad i_2^*(\alpha) = 1$

$$\Rightarrow \quad \pi_1(X) = \pi_1(A) * \pi_1(B) / \langle a \rangle$$

$$\cong \mathbb{Z}\langle a \rangle \oplus \mathbb{Z}\langle b \rangle / \langle a \rangle \cong \mathbb{Z}\langle b \rangle \cong \mathbb{Z}$$

4.) $H_p(X;R)$ for $X = S^1 \times S^1 \sqcup B^2 / \partial B^2 \sim S^1 \times \{x\}$.

$A = B^2$, $B = T^2$, $A \cap B = S^1$, $A \cup B = X$.

Mayer-Vietoris:

$$0 \to H_2(S^1) \to H_2(A) \oplus H_2(B) \to H_2(X) \to H_1(S^1)$$
$$\to H_1(A) \oplus H_1(B) \to H_1(X) \to H_0(S^1) \to H_0(A) \oplus H_0(B)$$
$$\to H_0(X) \to 0$$

X p.c. $\Rightarrow$ $H_0(X) \cong \mathbb{Z}$; #3 $\Rightarrow$ $H_1(X) \cong \mathbb{Z}$.

$$\Rightarrow \quad 0 \to \mathbb{Z} \xrightarrow{\psi} H_2(X) \xrightarrow{\phi} \mathbb{Z} \xrightarrow{\varepsilon} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\delta} \mathbb{Z} \xrightarrow{\gamma} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z} \oplus \mathbb{Z}$$
$$\xrightarrow{\alpha} \mathbb{Z} \to 0.$$

α surj. $\Rightarrow$ $\ker \alpha = \mathbb{Z}$ $\Rightarrow$ $\mathrm{im}\, \beta = \mathbb{Z}$ $\Rightarrow$ $\ker \beta = 0$ $\Rightarrow$ $\mathrm{im}\, \gamma = 0$

$\Rightarrow$ $\ker \gamma = \mathbb{Z}$ $\Rightarrow$ $\mathrm{im}\, \delta = \mathbb{Z}$ $\Rightarrow$ $\ker \delta = \mathbb{Z}$ $\Rightarrow$ $\mathrm{im}\, \varepsilon = \mathbb{Z}$ $\Rightarrow$ $\ker \varepsilon = 0$

$\Rightarrow$ $\mathrm{im}\, \phi = 0$

and ψ is injective, hence $\mathrm{im}\, \psi = \mathbb{Z}$ $\Rightarrow$ $\ker \phi = \mathbb{Z}$

S

$$\underline{H_2(X) = \mathbb{Z}}$$

⑤ T/F: surj. $p: \tilde{X} \to X$

p covering $\Leftrightarrow$ p local diffeo

✗ $\rightsquigarrow$ consider $p: (0,2) \longrightarrow S^1$

$t \longmapsto e^{2\pi i t}$

$1 \in S^1$ is not evenly covered.

⑥ X p.c., $\pi_1(X) = 1$, $\pi_2(X) = 1$.

Consider T^2; show every $f: T^2 \to X$ is homotopic to const.

$T^2 =$  $\longrightarrow X$ $\rightsquigarrow$ restrict to edge, ~~map~~ maps homotopic to const,

$\| $

$S^1 \times S^1$

hence $f \simeq f'$ sending edges to const.

~~$f(s,t) = (\alpha(s), \beta(t))$~~

$\pi_2(X) = 1 \Rightarrow$ every $\alpha: I \times I \to X$ $\exists$ α_t w/ $\alpha_0 = \alpha$, $\alpha_1 = x_0$

$f \simeq [f] \in \pi_2(X, x_0)$

$\Rightarrow f \simeq$ const.