Geometry/Topology qualifying exam
Spring 1999

1. Let M be an embedded compact surface in $\mathbb{R}^3$, namely a non-empty 2–dimensional submanifold of $\mathbb{R}^3$. Show that there exists an infinite number of vertical lines $L = \{x\} \times \{y\} \times \mathbb{R}$ which meet M and are not tangent to it in the following sense: $M \cap L$ is non-empty and, for every $x \in M \cap L$, the plane tangent to M at x is not vertical.

2. Let N be a n–dimensional submanifold of the m–dimensional manifold M, and let $i : N \to M$ be the inclusion map. Suppose that N is closed in M. Show that, if $\alpha \in \Omega^p(N)$ is a degree p differential form on N, there exists a form $\beta \in \Omega^p(M)$ on M such that $i^*(\beta) = \alpha$. If $d\alpha = 0$, can you always choose β so that $d\beta = 0$?

3. In $B^2 \times B^2$, let X be the union of the torus $S^1 \times S^1$ and of the disk $B^2 \times \{x\}$ (where B^2 is the closed unit disk in $\mathbb{R}^2$ and S^1 is its boundary circle). Compute the fundamental group of X.

4. For X as in Problem 3, compute the homology modules $H_p(X; R)$, with coefficients in an arbitrary unitary ring R.

5. Prove or disprove: A surjective map $p : \widetilde{X} \to X$ is a covering map if and only if, for every $\widetilde{x} \in \widetilde{X}$, there is a neighborhood $\widetilde{U}$ of $\widetilde{x}$ such that the restriction $p_{|\widetilde{U}} : \widetilde{U} \to p\left(\widetilde{U}\right)$ is a homeomorphism.

6. Let X be a path connected space such that $\pi_1(X; x_0) = 1$ and $\pi_2(X; x_0) = 1$. Recall that the second property means that, for every continuous map $\alpha : [0,1] \times [0,1] \to X$ such that $\alpha(s,t) = x_0$ when $s \in \{0,1\}$ or $t \in \{0,1\}$, there is a homotopy $H : [0,1] \times [0,1] \times [0,1] \to X$ such that $H(s,t,u) = x_0$ when $s \in \{0,1\}$ or $t \in \{0,1\}$ or $u = 1$. Consider the 2–dimensional torus $T^2 = S^1 \times S^1$. Show that every continuous map $f : T^2 \to X$ is homotopic to a constant map. (Possible hint: Write the torus as a square with identifications of its sides.)