Geometry/Topology Exam, Spring 1998

1) Let $X = S^1 \times S^1 - \{x, y\}$, where the two points $x, y \in S^1 \times S^1$ are distinct. Compute the fundamental group $\pi_1(X; x_0)$ and the homology R–modules $H_n(X)$.

2) Let $f : S^n \to S^n$ be a continuous map such that $H_n(f) : H_n(S^n) \to H_n(S^n)$ is non-trivial. Show that f is surjective.

3) Let B^n be the unit ball in $\mathbf{R}^n$, with boundary the $n-1$–sphere S^{n-1}. If $f : B^n \to \mathbf{R}^n$ is a continuous map with $f(S^{n-1}) \subset B^n$ (but not necessarily $f(B^n) \subset B^n$), show that there exists an $x \in B^n$ such that $f(x) = x$.

4) Let $f : M \to M$ be a diffeomorphism of the manifold M so that $f^n = Id$, but $f, f^2, \ldots, f^{n-1}$ have no fixed point. Let M/f denote the quotient space of M by the equivalence relation which identifies $x, y \in M$ when there exists p with $y = f^p(x)$.

a) Show that M/f is a manifold.

b) Compute $\pi_1(M/f)$ if M is simply connected.

5) Consider the map $\phi(x, y, z) = (x^2 - y^2, xy, xz, yz)$.

a) Show that the image of the unit sphere $S^2 \subset \mathbf{R}^3$ (of equation $x^2 + y^2 + z^2 = 1$) is a submanifold of $\mathbf{R}^4$.

b) Show this image $\phi(S)$ is diffeomorphic to the real projective plane $\mathbf{RP}^2$.

6) Calculate the integral

$$\int_{S^2} \omega$$

where S^2 is the standard unit sphere in $\mathbf{R}^3$, and ω is the 2–form

$$\omega = (x^2 + xy)(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy).$$