Qualifying Exam in Geometry/Topology - May 1997

Problem 1. (a) Define the *degree* of a smooth map between smooth, compact, oriented manifolds of the same dimension. (b) Suppose that M is a smooth, compact, oriented manifold of dimension $m \geq 1$. Prove that there exists a smooth map $f: M \to S^m$ of degree 1. *(Note: Here S^m denotes the standard m-sphere.)*

Problem 2. Using Sard's Theorem together with the classification theorem for one-dimensional manifolds with boundary, prove that there does not exist a smooth retraction from $D^n \to S^{n-1}$. *(Note: Here D^n denotes the closed unit disk and S^{n-1} denotes its boundary, the unit sphere. By a retraction we mean a map $D^n \to S^{n-1}$ whose restriction to $S^{n-1} \subset D^n$ is the identity map.)*

Problem 3. Does there exist a curve segment C in the standard two-sphere $S^2 \subseteq R^3$ running from the South Pole to the North Pole and meeting each latitude (i.e., each level set $z =$ constant) at an angle of $\pi/4$?

Problem 4. Let U be an open subset of $\mathbf{R}^n$, and let $\omega \in \Omega^p(U \times \mathbf{R})$ be a differential form of degree p on $U \times \mathbf{R}$. *(Remark: Here $\Omega^p(X)$ denotes the space of differential forms of degree p on X.)*

1. Show that there exists a family of differential forms $\alpha_t \in \Omega^p(U)$ and $\beta_t \in \Omega^{p-1}(U)$, depending differentiably on t, such that

$$\omega(x,t) = \alpha_t(x) + \beta_t(x) \wedge dt \tag{1}$$

   at every $(x,t) \in U \times \mathbf{R}$.

2. Show that there exists $\omega' \in \Omega^{p-1}(U \times \mathbf{R})$ such that $\omega = d\omega'$ if and only if $d\omega = 0$ and there exists $\alpha' \in \Omega^{p-1}(U)$ such that $\alpha_7 = d\alpha'$.

Problem 5. Compute the homology groups $H_n(S^2 \times S^q; \mathbf{Z})$, where S^p denotes the p-dimensional sphere.

Problem 6. Recall that two covering spaces $p : \tilde{X} \to X$ and $p' : \tilde{X}' \to X$ over the same space X are *isomorphic* if there is a homeomorphism $\phi : \tilde{X} \to \tilde{X}'$ such that $p' \circ \phi = p$. Let X be the Klein bottle $S^1 \times [0,1]/\sim$ where $\sim$ identifies $(z, 0)$ to $(\overline{z}, 1)$. Up to isomorphism, how many coverings $p : \tilde{X} \to X$ with $p^{-1}($ one point $) =$ three points are there? *(Hint: Consider monodromy.)*