Qualifying Exam in Topology and Geometry – Spring 1996

Directions: Do six of the following seven problems. Show clearly all of your work.

Problem 1. (a) State carefully the classification of closed, compact, connected, oriented topological surfaces without boundary. That is, describe a list of such surfaces so that any other such surface is homeomorphic to exactly one surface on your list. Briefly describe the proof. (b) Extend this result to give a classification of closed, compact, connected, oriented topological surfaces with boundary. (c) We say that two simple closed curves C and D in an orientable topological surface M are *equivalent* if there is an orientation-preserving homeomorphism $f : M \to M$ such that $f(C) = D$. Give a complete list of all equivalence classes of curves in a closed, compact, orientable topological surface M without boundary.

Problem 2. Let S^m be the m-dimensional sphere, and let M^m be a smooth compact oriented m-dimensional manifold. Suppose that $f : S^m \to M^m$ is a smooth map of degree one. Prove that M is a cohomology m-spere (i.e. has the same cohomology groups as S^m).

Problem 3. Let $M \subset \mathbf{R}^3$ be a smooth compact surface with constant Gaussian curvature. (a) Explain why M must be diffeomorphic to a two-sphere. (b) Prove that in fact M is a Euclidean two-sphere.

Problem 4. Let $a, b \in \mathbf{C}$ be two points in the complex plane. Assume that

$$\int_a^b z^5 dz = 0 = \int_a^b z^{46} dz.$$

Prove that $a = b$.

Problem 5. Consider the subset $M \subset \mathbf{R}^3$ defined by $x^{30} + y^{30} + z^{30} = 1$. Prove that M is a smooth surface, and compute the integral of $x^{15}y^{14}z^{14}dy \wedge dz$ over M.

Problem 6. Contruct a topological space whose fundamental group is isomorphic to the group $< a, b \mid a^2b^3 = 1 >$.

Problem 7. Let $f(z)$ be a complex polynomial of degree 5. Recall that the extended complex plane $\mathbf{C} \cup \{\infty\}$ can be identified with the two-sphere S^2 via stereographic projection. Think of $f(z)$ as being a map $f : S^2 \rightarrow S^2$. Compute the map of homology groups $f_* : H_*(S^2) \rightarrow H_*(S^2)$.