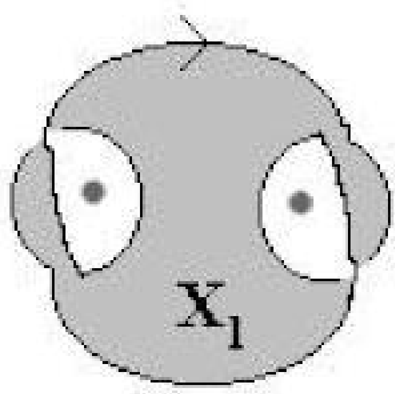
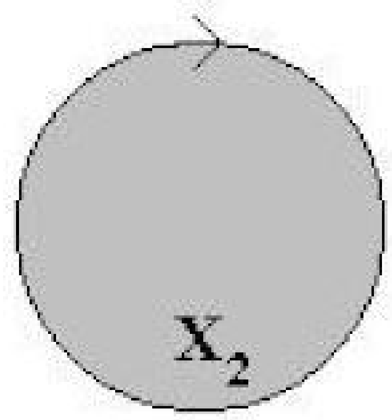
GEOMETRY TOPOLOGY QUALIFYING EXAM (MATH 535A AND MATH 540)

SPRING 1995

Problem 1 Show that the tangent bundle of a differentiable manifold is an oriented manifold.

Problem 2 Let X_1 be the "double Möbius strip" shown below. Let X_2 be the disk. Note that the boundaries ∂X_1 and ∂X_2 of these two surfaces are both homoeomorphic to a circle. Choose a homeomorphism $\varphi : \partial X_1 \to \partial X_2$ and let $X = X_1 \cup X_2 / \sim$, where the equivalence relation $\sim$ identifies each $x_1 \in X_1$ to $\varphi(x_1) \in \partial X_2$. (In other words, X is obtained by gluing X_1 and X_2 along their boundary.) Give a presentation for the fundamental group of X.





Problem 3 Let X be as in Problem 2. Compute all the holomogy groups $H_n(X, \mathbb{Z})$.

Problem 4 Let S^2 be the two-dimensional sphere and let T^2 be the two-dimensional torus. Prove that, for every continuous mapping $f : S^2 \to T^2$, the induced map in homology $H_2(f) : H_2(S^2, \mathbb{R}) \to H_2(T^2, \mathbb{R})$ is zero.

Problem 5 Let $M \subset \mathbb{R}^3$ be the subset defined by $x^6 + y^6 + z^6 = 1$. Prove that M is a smooth submanifold of $\mathbb{R}^3$ and compute the integral of $x^3y^2z^2 \, \mathrm{d}y \wedge \, \mathrm{d}z$ over M.

Problem 6 Two coverings $p : \tilde{X} \to X$ and $p' : \tilde{X}' \to X$ are said to be equivalent if there is a homomorphism $\varphi : \tilde{X} \to \tilde{X}'$ such that $p' \circ \varphi = p$. If X is the figure eight ∞, how many equivalence classes of coverings $p : \tilde{X} \to X$ with $p^{-1}(x) =$(three points) are there?

Problem 7 Use differential geometry to prove the Cauchy Integral Theorem:

If $f : \Omega \to \mathbb{C}$ is a holomorphic function on an open subset Ω of the complex plane, and if $c : [0, 1] \to \Omega$ is a differentiable curve iwth $c(0) = c(1)$ and $[c] = 0$ in $H_1(\Omega, \mathbb{Z})$, then $\int_c f(z) \, \mathrm{d}z = 0$.