GEOMETRY TOPOLOGY QUALIFYING EXAM (MATH 535A AND MATH 540)

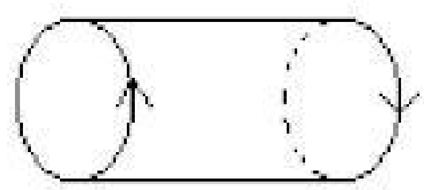
SPRING 1994

Problem 1 Let $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$. Does there exist a submersion $f : S^2 \to \mathbb{R}^2$, namely a map such that the tangent map $T_x f : T_x S^2 \to \mathbb{R}^2$ is everywhere surjective?

Problem 2 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable function and let $M = f^{-1}(0)$. Assume that the tangent map $R_x M : \mathbb{R}^n \to \mathbb{R}$ is non-trivial at each $x \in M$. Is M necessarily a manifold? Is M necessarily orientable? Give a proof or a counterexample.

Problem 3 Let $f : \mathbb{C} - \{-1, 0, 1\} \to \mathbb{C} - \{0, 1\}$ be defined by $f(z) = z^2$. Show that the homomorphism $f_* : \pi_1(\mathbb{C} - \{-1, 0, 1\}; 2) \to \pi_1(\mathbb{C} - \{0, 1\}; 4)$ is injective. Compute the groups $\pi_1(\mathbb{C} - \{-1, 0, 1\}; 2)$ and $\pi_1(\mathbb{C} - \{0, 1\}; 4)$ and determine the homomorphism f_*.

Problem 4 Compute the fundamental group of the Klein bottle. (See figure below.)



Problem 5 Let $B_1, \ldots, B_p$ be p disjoint copies of the n-dimensional closed ball B^n, and let X be the space obtained by gluing these balls along their boundary. Namely, choose a homeomorphism $\varphi_i : B^n \to B_i$ for every i. Then, X is the quotient of the space $\cup_{i=1}^p B_i$ by the equivalence relation whose equivalence classes are all $\{x\}$ with x in the interior of some B_i as well as all subsets $\{\varphi_1(y), \varphi_2(y), \ldots, \varphi_p(y)\}$ with $y \in S^n$. Compute the homology groups of X.

Problem 6 Let ω be closed differential form of degree 1 defined on $\mathbb{R}^3 - L$, where L is a subset shown below (made up of the z-axis, the unit circle and a half line in the xy-plane). Let γ be the closed curve shown. Calculate $\int_\gamma i^*(\omega)$, where $i : \gamma \to \mathbb{R}^3 - L$ is the inclusion map. (Hint: Be smart, apply Stokes to a suitably chosen surface).