GEOMETRY TOPOLOGY QUALIFYING EXAM (MATH 535A AND MATH 540)

SPRING 1993

Problem 1 Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be differentiable. Compute $f^*(\, dx_1 \wedge \, dx_2)$.

Problem 2 Let $f : M \to N$ be a differentiable map between two manifolds, such that f is bijective and such that its tangent map $T_x f : T_x M \to T_{f(x)} N$ is an isomophism for every $x \in M$. Show that f is a diffeomorphism.

Problem 3 Let $B^2 = \{x \in \mathbb{R}^2; ||x|| \leq 1\}$ be the unit disk in the plane. Let $f : B^2 \to B^2$ be a continuous map such that $f(x) = x$ for every $x \in S^1 = \{x \in \mathbb{R}^2; ||x|| = 1\}$. Show that f is surjective.

Problem 4 Let M be a compact surface in $\mathbb{R}^3$, namely a compact 2-dimensional submanifold of $\mathbb{R}^3$. Show that there is a point $x \in M$ s uch that M lies entirely on one side of the tangent plane $T_x M$.

Problem 5 Is there a covering map $\mathbb{R}^2$-{2 points}$\to \mathbb{R}^2$-{1 point}? (Possible hint: π_1 and H_1).

Problem 6 Let U be an open subset of $\mathbb{R}^n$. Show that U is homeomorphic to no open subset of $\mathbb{R}^p$ with $p < n$. (Possible hint: consider the homology of a pair $(U, U - \{x\})$).

Problem 7 Recall that the tangent bundle TM of a manifold M consists of all pairs $(x, \vec{v})$ where $x \in M$ and $\vec{v}$ is the tangent space $T_x M$ of M at x. Show that TM is an oriented manifold (even when M is not orientable!).