Topology Qualifying Exam Spring 2025
Solutions and Notes

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Problem 1

Let X be the quotient $S^2/\sim$ by the relation

$$(x, y, z) \sim (x', y', z') \qquad \text{if and only if} \qquad z = z' = 0 \text{ and } (x, y) = \pm(x', y').$$

In other words, X is acquired by S^2 by quotienting the equator $S^1 \subset S^2$ by the map $v \to -v$.

(a) Describe a cell decomposition of X. That is, describe the cells and the attaching maps.

(b) Give the corresponding cellular homology complex (with $\mathbb{Z}$-coefficients) for the cell decomposition in (a).

(c) Compute the homology of the cellular complex from (b).

Solution:

(a) Cell decomposition.

Intuition.

- The quotient identifies only antipodal points on the *equator* $S^1 \subset S^2$. Thus the equator becomes
$$S^1/(\pm 1) \cong \mathbb{RP}^1 \cong S^1,$$
with one 0-cell e^0 and one 1-cell e^1.
- The upper and lower open hemispheres map to two 2-cells in X, denoted e^2_+ and e^2_-.
- Before the quotient, each boundary $\partial D^2_\pm \cong S^1$ goes once around the equator (degree $+1$ for the top, -1 for the bottom). After identifying antipodal points, going once around the equator means you traverse each identified pair exactly once, so the image goes twice around the quotient circle. Hence the attaching maps have degrees $+2$ and -2 onto the 1-cell e^1.

Cells.

- e^0: one single 0-cell (choose a basepoint on the quotient equator).
- e^1: the equator modulo antipodes is $\mathbb{RP}^1 \cong S^1$; give it the usual CW with one 0-cell and one 1-cell. Thus we have one 1-cell e^1.

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- e^2_+, e^2_-: the two hemispheres of S^2 descend to two 2-cells.

Attaching maps. The boundary of each hemisphere $\partial D^2 \cong S^1$ first maps to the equator by the identity (orientation is opposite for the two hemispheres in S^2), then to $\mathbb{RP}^1$ by the quotient $p : S^1 \to S^1/(\pm 1)$, which has degree 2. Hence the two attaching maps $S^1 \to e^1$ have degrees $+2$ and -2.

(b) Cellular complex. The cellular chain complex (with $\mathbb{Z}$–coefficients) is

$$0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0,$$

with the chosen cellular bases (by (a) *Cells*)

$$C_2 = \mathbb{Z}\langle e^2_+, e^2_-\rangle \cong \mathbb{Z}^2, \qquad C_1 = \mathbb{Z}\langle e^1\rangle \cong \mathbb{Z}, \qquad C_0 = \mathbb{Z}\langle e^0\rangle \cong \mathbb{Z}.$$

The differentials are (by (a) *Attaching maps*)

$$\partial_1 = 0, \qquad \partial_2(e^2_+) = 2\,e^1, \quad \partial_2(e^2_-) = -2\,e^1,$$

so for $(a, b) \in \mathbb{Z}^2 \cong C_2$,

$$\partial_2(a, b) = (2a - 2b)\,e^1.$$

With respect to the ordered bases $\{e^2_+, e^2_-\}$ of C_2 and $\{e^1\}$ of C_1, the matrix of ∂_2 is

$$[\partial_2] = \begin{bmatrix} 2 & -2 \end{bmatrix}.$$

Hence the full chain complex is

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 2 & -2 \end{bmatrix}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0.$$

(c) Homology.

Recall (homology from the chain complex).

For a chain complex

$$0 \to C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0,$$

the homology groups are

$$H_2 = \ker \partial_2, \qquad H_1 = \ker \partial_1 / \operatorname{im} \partial_2, \qquad H_0 \cong \mathbb{Z}^{\#\text{path components of } X}.$$

Compute H_2.

$$\ker \partial_2 = \{(a, b) \in \mathbb{Z}^2 : \; 2a - 2b = 0\} = \{(t, t) : t \in \mathbb{Z}\} = \mathbb{Z} \cdot (1, 1) \cong \mathbb{Z}.$$

Hence

$$H_2(X) = \ker \partial_2 \cong \mathbb{Z}, \quad \text{generated by } [z_2] \text{ with } z_2 = (1, 1) = e^2_+ + e^2_-.$$

(In words: the sum of the two 2-cells is a 2-cycle.)

Compute H_1. Since $\partial_1 = 0$, $\ker \partial_1 = C_1 = \mathbb{Z}\langle e^1\rangle$. The image of ∂_2 is

$$\operatorname{im} \partial_2 = \{(2a - 2b)e^1 : \; a, b \in \mathbb{Z}\} = 2\mathbb{Z}\langle e^1\rangle \cong 2\mathbb{Z}.$$

Therefore

$$H_1(X) = \ker \partial_1 / \operatorname{im} \partial_2 \cong \mathbb{Z}\langle e^1\rangle / 2\mathbb{Z}\langle e^1\rangle \cong \mathbb{Z}/2\mathbb{Z},$$

generated by the class $[e^1]$ with relation $2[e^1] = 0$.

Compute H_0. Recall that $H_0(X;\mathbb{Z}) \cong \bigoplus_{\pi_0(X)} \mathbb{Z}$ (one copy of $\mathbb{Z}$ for each path component of X). Since X is connected, $H_0(X) \cong \mathbb{Z}$.

Summary.

$$H_k(X) \;\cong\; \begin{cases} \mathbb{Z}, & k=0,\ \text{generated by } [e^0], \\ \mathbb{Z}/2\mathbb{Z}, & k=1,\ \text{generated by } [e^1] \text{ (with } 2[e^1]=0), \\ \mathbb{Z}, & k=2,\ \text{generated by } [e^2_+ + e^2_-], \\ 0, & k \geq 3. \end{cases}$$

Remark (intuition). The data above identify $X \simeq S^2 \vee \mathbb{RP}^2$: the kernel direction $(1,1)$ yields an S^2, and the degree-2 attachment yields $\mathbb{RP}^2$.

Problem 2

Consider the space $Y = \mathbb{RP}^3 \vee (S^2 \times S^1)$.

(a) Compute the homology groups (with $\mathbb{Z}$ coefficients) of Y.

(b) Is Y homotopy equivalent to a compact orientable manifold? Prove or disprove.

(c) Compute the fundamental group $\pi_1(Y)$.

(d) Compute the Euler characteristic $\chi(Y)$.

Solution:

(a) Homology groups.

Recall.

For connected CW spaces, $\widetilde{H}_k(A \vee B) \cong \widetilde{H}_k(A) \oplus \widetilde{H}_k(B)$. Also

$$H_*(\mathbb{RP}^3): \; H_0 = \mathbb{Z}, \; H_1 = \mathbb{Z}/2, \; H_2 = 0, \; H_3 = \mathbb{Z},$$
$$H_*(S^2 \times S^1): \; H_0 = \mathbb{Z}, \; H_1 = \mathbb{Z}, \; H_2 = \mathbb{Z}, \; H_3 = \mathbb{Z}.$$

Therefore

$$H_k(Y) \cong \begin{cases} \mathbb{Z}, & k = 0, \\ \mathbb{Z}/2 \; \oplus \; \mathbb{Z}, & k = 1, \\ \mathbb{Z}, & k = 2, \\ \mathbb{Z} \oplus \mathbb{Z}, & k = 3, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Manifold test (Poincaré duality).

Recall.

Top homology of a compact n–manifold (by Poincaré duality): Let M be a compact, connected n–manifold and R a commutative ring; then

$$H_n(M; R) = \begin{cases} R, & \partial M = \varnothing \text{ and } M \text{ orientable over } R, \\ 0, & \text{otherwise,} \end{cases}$$

(Note that over $\mathbb{Z}_2$, every manifold is orientable.)

Thus, if M is a compact orientable 3-manifold, then

$$H_3(M) \cong \begin{cases} \mathbb{Z}, & \partial M = \varnothing, \\ 0, & \partial M \neq \varnothing. \end{cases}$$

But from (a), $H_3(Y) \cong \mathbb{Z} \oplus \mathbb{Z}$, which is impossible in either case. Hence Y is *not* homotopy equivalent to a compact orientable 3-manifold.

(c) Fundamental group.

Recall (van Kampen for wedges).

If A and B are path-connected spaces and $A \vee B$ is their wedge (one point identified), then

$$\pi_1(A \vee B) \;\cong\; \pi_1(A) * \pi_1(B),$$

the free product of the fundamental groups.

In our case,

$$\pi_1(\mathbb{RP}^3) \cong \mathbb{Z}/2, \qquad \pi_1(S^2 \times S^1) \cong \pi_1(S^2) \times \pi_1(S^1) \cong 0 \times \mathbb{Z} \cong \mathbb{Z}.$$

Therefore

$$\pi_1(Y) \;=\; \pi_1\big(\mathbb{RP}^3 \vee (S^2 \times S^1)\big) \;\cong\; (\mathbb{Z}/2) * \mathbb{Z}.$$

(d) Euler characteristic.

Recall (Euler characteristic).

- For a wedge of connected CW complexes,

$$\chi(A \vee B) = \chi(A) + \chi(B) - 1.$$

- For finite CW complexes, the Euler characteristic is multiplicative on products:

$$\chi(A \times B) = \chi(A)\,\chi(B).$$

- In particular,

$$\chi(S^n) = \begin{cases} 2, & n \text{ even}, \\ 0, & n \text{ odd}, \end{cases} \qquad \chi(\mathbb{RP}^3) = 0.$$

Using these facts,

$$\chi(S^2 \times S^1) = \chi(S^2)\chi(S^1) = 2 \cdot 0 = 0, \qquad \chi(\mathbb{RP}^3) = 0.$$

Since $Y = \mathbb{RP}^3 \vee (S^2 \times S^1)$, we get

$$\chi(Y) = \chi(\mathbb{RP}^3) + \chi(S^2 \times S^1) - 1 = 0 + 0 - 1 = -1.$$

Problem 3

(a) Let ΣX denote the suspension of a connected topological space X.[1] Using the Mayer–Vietoris sequence, compute the integral homology groups of ΣX in terms of those of X.

(b) Prove that there is an isomorphism $\pi_7(S^3) \cong \pi_7(S^2)$.

Solution:

(a) Homology of the suspension.

Recall (Mayer–Vietoris).

If a space Y is covered by open sets U, V with $Y = U \cup V$, there is a (reduced) long exact sequence

$$\cdots \longrightarrow \widetilde{H}_k(U \cap V) \longrightarrow \widetilde{H}_k(U) \oplus \widetilde{H}_k(V) \longrightarrow \widetilde{H}_k(Y) \longrightarrow \widetilde{H}_{k-1}(U \cap V) \longrightarrow \cdots.$$

We will apply this with $Y = \Sigma X$ and a suitable cover U, V.

We write ΣX as a union of two open sets:

$$U = \text{open cone on the "top" of } X, \qquad V = \text{open cone on the "bottom" of } X.$$

Concretely, U is a cone from the north suspension point and V from the south suspension point. Each is contractible:

$$U \simeq * \simeq V \implies \widetilde{H}_k(U) = \widetilde{H}_k(V) = 0 \text{ for all } k.$$

Their intersection is the "cylinder region" between the two tips:

$$U \cap V \cong X \times (0,1) \;\simeq\; X,$$

so $\widetilde{H}_k(U \cap V) \cong \widetilde{H}_k(X)$ for all k.

Now apply the reduced Mayer–Vietoris sequence for $\Sigma X = U \cup V$:

$$\cdots \to \widetilde{H}_k(U\cap V) \xrightarrow{\alpha_k} \widetilde{H}_k(U)\oplus\widetilde{H}_k(V) \to \widetilde{H}_k(\Sigma X) \to \widetilde{H}_{k-1}(U\cap V) \xrightarrow{\alpha_{k-1}} \widetilde{H}_{k-1}(U)\oplus\widetilde{H}_{k-1}(V) \to \cdots$$

But $\widetilde{H}_k(U) = \widetilde{H}_k(V) = 0$ for all k, so the sequence at level k simplifies to

$$\widetilde{H}_k(U \cap V) \longrightarrow 0 \longrightarrow \widetilde{H}_k(\Sigma X) \longrightarrow \widetilde{H}_{k-1}(U \cap V) \longrightarrow 0.$$

Exactness gives:

$$\widetilde{H}_k(\Sigma X) \;\cong\; \widetilde{H}_{k-1}(U \cap V) \;\cong\; \widetilde{H}_{k-1}(X) \quad \text{for all } k.$$

Finally, since ΣX is connected (for connected X), we have

$$H_0(\Sigma X) \cong \mathbb{Z}.$$

So the answer can be summarized as

$$\widetilde{H}_k(\Sigma X) \cong \widetilde{H}_{k-1}(X) \text{ for all } k, \quad H_0(\Sigma X) \cong \mathbb{Z}.$$

[1] Recall that the suspension of a topological space X is given by $\Sigma X := (X \times I)/\sim$, where we put $(x, 0) \sim (y, 0)$ and $(x, 1) \sim (y, 1)$ for all $x, y \in X$.

(b) $\pi_7(S^3) \cong \pi_7(S^2)$**.**

Recall (Fibration and LES).

If $F \hookrightarrow E \xrightarrow{p} B$ is a (Serre) fibration, then there is a long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_k(F) \longrightarrow \pi_k(E) \longrightarrow \pi_k(B) \longrightarrow \pi_{k-1}(F) \longrightarrow \cdots.$$

In particular, the *Hopf fibration* $S^1 \hookrightarrow S^3 \to S^2$ is such a fibration, and we also know $\pi_k(S^1) = 0$ for all $k \geq 2$. More generally, for $n \geq 1$,

$$\pi_k(S^n) = \begin{cases} 0, & 0 < k < n, \\ \mathbb{Z}, & k = n, \end{cases}$$

while the groups $\pi_k(S^n)$ for $k > n$ are complicated and not given by a simple closed formula.

Use the Hopf fibration

$$S^1 \hookrightarrow S^3 \xrightarrow{h} S^2.$$

The associated long exact sequence of homotopy groups contains the segment

$$\cdots \longrightarrow \pi_7(S^1) \longrightarrow \pi_7(S^3) \xrightarrow{h_*} \pi_7(S^2) \longrightarrow \pi_6(S^1) \longrightarrow \cdots.$$

For the circle, we know $\pi_k(S^1) = 0$ for all $k \geq 2$. In particular,

$$\pi_7(S^1) = 0 = \pi_6(S^1).$$

Therefore the relevant piece of the long exact sequence reduces to

$$0 \longrightarrow \pi_7(S^3) \xrightarrow{h_*} \pi_7(S^2) \longrightarrow 0,$$

so h_* is an isomorphism. Hence

$$\boxed{\pi_7(S^3) \cong \pi_7(S^2).}$$

Problem 4

(a) Let X_n be the complement of n distinct lines through the origin in $\mathbb{R}^3$. Compute $\pi_1(X_n)$.

(b) Give a pair of path-connected topological spaces X and Y such that $H_1(X)$ and $H_1(Y)$ are isomorphic, but $\pi_1(X)$ and $\pi_1(Y)$ are not.

(c) Compute the fundamental group of a genus g surfaces with n points removed for $g, n > 0$.

(d) Let X be a path-connected space such that $\pi_1(X) \cong \mathbb{Z}/7\mathbb{Z}$. Determine the set of integers n such that there exists a 9-sheeted covering space $p : \widetilde{X} \to X$ with n connected components.

Solution:

(a) Complement of lines.

Recall.

A deformation retraction $Y \to Z$ is a homotopy equivalence, so $\pi_1(Y) \cong \pi_1(Z)$. Also, for $k \geq 1$ the sphere with k points removed is homotopy equivalent to a wedge of $k-1$ circles, hence

$$\pi_1(S^2 \setminus \{k \text{ points}\}) \cong F_{k-1}$$

(the free group on $k-1$ generators).

Now consider

$$X_n \;=\; \mathbb{R}^3 \setminus \bigcup_{i=1}^{n} \ell_i,$$

where each ℓ_i is a line through the origin. Radial projection $\mathbb{R}^3 \setminus \{0\} \to S^2$ restricts to a deformation retraction

$$X_n \;\simeq\; S^2 \setminus \{2n \text{ points}\},$$

since each line meets S^2 in an antipodal pair. Therefore

$$\pi_1(X_n) \;\cong\; \pi_1(S^2 \setminus \{2n \text{ points}\}) \;\cong\; F_{2n-1},$$

the free group of rank $2n-1$.

(b) Same H_1 but different π_1. Take $X = \mathbb{T}^2 = S^1 \times S^1$ and $Y = S^1 \vee S^1$. Then

$$H_1(X) \cong \mathbb{Z}^2 \cong H_1(Y),$$

but

$$\pi_1(X) \cong \mathbb{Z}^2 \quad \text{(abelian)}, \qquad \pi_1(Y) \cong F_2 \quad \text{(nonabelian)},$$

so $\pi_1(X) \ncong \pi_1(Y)$.

Another example (higher genus). Let $X = \Sigma_g$ be a closed oriented surface of genus $g \geq 2$, and let $Y = \bigvee_{i=1}^{2g} S^1$ be a wedge of $2g$ circles. Then

$$H_1(X) \cong \mathbb{Z}^{2g} \cong H_1(Y),$$

but

$$\pi_1(X) \cong \langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle \quad \text{(surface group with one relation)},$$

while

$$\pi_1(Y) \cong F_{2g} \quad \text{(free group on } 2g \text{ generators)},$$

so again $\pi_1(X) \ncong \pi_1(Y)$ even though H_1 agrees.

(c) Punctured genus-g surface.

Recall (Hatcher P. 51).

The closed oriented genus g surface Σ_g can be obtained from a $4g$-gon with edge word

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$$

by identifying edges in pairs. The loops corresponding to the edges give generators $a_1, b_1, \ldots, a_g, b_g$, and the single relation coming from the boundary word is

$$\prod_{i=1}^{g} [a_i, b_i] \;=\; 1,$$

so

$$\pi_1(\Sigma_g) \cong \Big\langle a_1, b_1, \ldots, a_g, b_g \;\Big|\; \prod_{i=1}^{g} [a_i, b_i] = 1 \Big\rangle.$$

Now remove $n > 0$ points $p_1, \ldots, p_n$ from Σ_g. Take small disjoint disks around each p_j, and denote by c_j a (positively oriented) loop going once around the boundary of the j-th disk. These loops become additional generators in $\pi_1(\Sigma_g \setminus \{p_1, \ldots, p_n\})$.

Thus in the polygon picture, we now have the edge loops a_i, b_i as before, together with the n small boundary loops $c_1, \ldots, c_n$. The boundary of the polygon and of the deleted disks together gives a single relation:

$$\prod_{i=1}^{g} [a_i, b_i] \cdot c_1 \cdots c_n \;=\; 1.$$

Thus

$$\pi_1(\Sigma_g \setminus \{p_1, \ldots, p_n\}) \;\cong\; \Big\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n \;\Big|\; \prod_{i=1}^{g} [a_i, b_i] \cdot c_1 \cdots c_n = 1 \Big\rangle.$$

Since there is exactly one relation, we can solve for one generator, say

$$c_n \;=\; \Big(\prod_{i=1}^{g} [a_i, b_i] \cdot c_1 \cdots c_{n-1} \Big)^{-1},$$

so the remaining $2g + (n-1)$ generators $a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_{n-1}$ are free. Hence

$$\pi_1(\Sigma_g \setminus \{p_1, \ldots, p_n\}) \;\cong\; F_{2g+n-1},$$

the free group of rank $2g + n - 1$.

(d) 9-sheeted coverings with $\pi_1(X) \cong \mathbb{Z}/7\mathbb{Z}$.

Recall (classification of coverings).

For a nice space X (path-connected, locally path-connected, semilocally simply connected) with fundamental group $G = \pi_1(X)$, connected covering spaces of X (up to isomorphism) correspond to subgroups $H \leq G$, and the number of sheets of the cover is the index $[G : H]$.

A general (possibly disconnected) covering is a disjoint union of connected covers $\widetilde{X}_i \to X$, and the total number of sheets is

$$\deg(p) \;=\; \sum_i [G : H_i],$$

where each H_i is the subgroup corresponding to the component $\widetilde{X}_i$.

Here $G \cong \mathbb{Z}/7\mathbb{Z}$, which is cyclic of prime order. Its only subgroups are G itself and the trivial subgroup $\{0\}$, with indices

$$[G:G] = 1, \qquad [G:\{0\}] = 7.$$

Thus any connected covering has degree either 1 or 7.

Now let $p : \widetilde{X} \to X$ be a 9-sheeted covering, possibly disconnected. Write $\widetilde{X}$ as a disjoint union of connected components $\widetilde{X}_i$, with corresponding subgroups $H_i \leq G$. Then

$$9 \;=\; \sum_i [G:H_i], \qquad [G:H_i] \in \{1,7\}.$$

Let m be the number of components of degree 7, and r the number of components of degree 1. Then

$$7m + r = 9, \qquad m, r \in \mathbb{Z}_{\geq 0}.$$

The only nonnegative integer solutions are

$$(m,r) = (1,2) \quad \text{or} \quad (m,r) = (0,9).$$

Hence the possible numbers of connected components are

$$n = m + r \in \{1+2,\ 0+9\} = \{3,9\}.$$

Therefore a 9-sheeted covering of X with $\pi_1(X) \cong \mathbb{Z}/7\mathbb{Z}$ can have only 3 or 9 connected components.