Topology qualifying exam **Spring** 2025

1. Let X be the quotient $S^2/\sim$ by the relation

$$(x,y,z) \sim (x',y',z') \qquad \text{if and only if} \qquad z = z' = 0 \text{ and } (x,y) = \pm(x',y')$$

   In other words, X is acquired by S^2 by quotienting the equator $S^1 \subset S^2$ by the map $v \to -v$.

   (a) Describe a cell decomposition of X. That is, describe the cells and the attaching maps.

   (b) Give the corresponding cellular homology complex (with $\mathbb{Z}$-coefficients) for the cell decomposition in (a).

   (c) Compute the homology of the cellular complex from (b).

2. Consider the space $Y = \mathbb{RP}^3 \vee (S^2 \times S^1)$.

   (a) Compute the homology groups (with $\mathbb{Z}$ coefficients) of Y.

   (b) Is Y homotopy equivalent to a compact orientable manifold? Prove or disprove.

   (c) Compute the fundamental group $\pi_1(Y)$

   (d) Compute the Euler characteristic $\chi(Y)$.

3. (a) Let ΣX denote the suspension of a connected topological space X.[1] Using the Mayer–Vietoris sequence, compute the integral homology groups of ΣX in terms of those of X.

   (b) Prove that there is an isomorphism $\pi_7(S^3) \cong \pi_7(S^2)$.

4. (a) Let X_n be the complement of n distinct lines through the origin in $\mathbb{R}^3$. Compute $\pi_1(X_n)$.

[1] Recall that the suspension of a topological space X is given by $\Sigma X := (X \times I)/\sim$, where we put $(x,0) \sim (y,0)$ and $(x,1) \sim (y,1)$ for all $x, y \in X$.

(b) Give a pair of path-connected topological spaces X and Y such that $H_1(X)$ and $H_1(Y)$ are isomorphic, but $\pi_1(X)$ and $\pi_1(Y)$ are not.

(c) Compute the fundamental group of a genus g surfaces with n points removed for $g, n > 0$.

(d) Let X be a path-connected space such that $\pi_1(X) \cong \mathbb{Z}/7\mathbb{Z}$. Determine the set of integers n such that there exists a 9-sheeted covering space $p : \widetilde{X} \to X$ with n connected components.