Spring 2023 Solutions

Problem 1

Let X be a Hausdorff topological space and let $\pi : \tilde{X} \to X$ be its universal cover, i.e. $\tilde{X}$ is path connected and simply connected and π is a covering map. Prove that if $\tilde{X}$ is compact then the fundamental group of X is finite.

proof:

Fall 2020 problem 1

$|\pi_1(X)|$ = degree of universal cover

Since p is a cover, for any $x \in X$ $\exists$ U open in X s.t. $p^{-1}(U) = \bigsqcup V_{i_x}$

$\tilde{X} = \hat{X} \setminus \{p^{-1}(x) \cup p^{-1}(U)$

this is an open cover for $\tilde{X}$ $\Rightarrow$ $\exists$ a finite subcover

$\Rightarrow p^{-1}(U) = \bigsqcup_{i=1}^{n} V_x^i$, n = degree of covering

So $|\pi_1(X, x)| = n$

Problem 4

Let T^2 denote the standard 2-torus and S^2 the standard 2-sphere. Let X be the space obtained by identifying 2 distinct points a_1, a_2 from T^2 to some point $p \in S^2$. Compute the integral homology groups of X and the fundamental group of X

proof:
T^2 with 2 points identified is homotopic to $T^2 \vee S^1$ by

> Torus with n points identified
>
> T^2 with n points identified is homotopic to the wedge sum of T^2 and $n-1$ circles.
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. Because these points are also identified on the sphere we have

$$X = T^2 \vee S^1 \vee S^2$$

By

> Reduced Homology of Wedge Sum
>
> **Hatcher Ch.2 Ex. 31**
> If the basepoints of X and Y that are identified in $X \vee Y$ are Deformation Retracts of neighborhoods $U \subset X$ and $V \subset Y$ then
>
> $$\tilde{H}_n(X \vee Y) \cong \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$$
>
> **Julian Take-home:**
> if (X, x) and (Y, y) are both good pairs, then
>
> $$\tilde{H}_n(X \vee Y) \simeq \tilde{H}_n(X) \oplus \tilde{H}_n(Y)$$
>
> # Proof
>
> Can be shown by mayer-vietoris
>
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we have

$$H_n(X) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} & n = 2 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & n = 3 \\ 0 & \text{else} \end{cases}$$

The fundamental group can be calculated via

> Seifert-Van Kampen
>
> **Rotman: Corollary 7.42**
> If K is a simplicial complex, having connected subcomplexes L_1 and L_2 such that $L_1 \cup L_2 = K$ and $L_1 \cap L_2$ is simply connected, then for $v_0 \in \text{Vert}(L_1 \cap L_2)$
>
> $$\pi(K, v_0) \cong \pi(L_1, v_0) * \pi(L_2, v_0)$$
>
> **Hatcher: Theorem 1.20**
> If X is the union of path-connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path-connected, then the homomorphism $\Phi : *_a \pi_1(A_\alpha) \to \pi_1(X)$ is surjective. If in addition each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Φ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta}(\omega) i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$ and hence Φ induces an isomorphism $\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)/N$
>
> **Andrews University (simply connected intersection):**
> If $X = A \cup B$ where A, B open, path connected and $A \cap B$ is simply connected then
>
> $$\pi_1(X) \cong \pi_1(A) * \pi_1(B)$$
>
> **Andrews University (general version):**

If $X = A \cup B$ where A, B open, path connected and $A \cap B$ is path-connected then

$$\pi_1(X) \cong \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)$$

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so we get

$$\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}^2 * 0 \cong \mathbb{Z} * \mathbb{Z}^2$$

Problem 6

Let X be a topological space. Suppose for some k that we can cover X by k open sets $U_1, \dots, U_k$ so that each U_i is contractible as is each higher intersection of s open sets $Ui_i \cap \dots \cap Ui_s$ for every s. Proved that the reduced homology $\tilde{H}_i(X) = 0$ for all $i \geq k-1$

Use induction on

Mayer-Vietoris

Julian Take-home Midterm:
If $U, V \subset X$ are subsets with $U^o \cup V^o = X$ then there is a long exact sequence

$$\dots \to H_n(U \cap V) \overset{(i_*, j_*)}{\to} H_n(U) \oplus H_n(V) \overset{k_* - l_*}{\to} H_n(X) \overset{\partial}{\to} H_{n-1}(U \cap V) \to \dots$$

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