2017, Spring

Problem 1.

Background. A *symplectic manifold* is a pair (M^{2n}, ω) consisting of an even-dimensional manifold M together with a closed nondegenerate 2-form $\omega \in \Omega^2(M)$. It follows from what we prove in this problem that an *exact symplectic manifold*, that is, a symplectic manifold (M, ω) with ω exact, also has exact *symplectic volume form* $\omega^{\wedge n}$.

Since $\mathsf{d}\omega = 0$, we have $\mathsf{d}(\alpha \wedge \underbrace{\omega \wedge \ldots \wedge \omega}_{(n-1)\text{ times}}) = (\mathsf{d}\alpha) \wedge \underbrace{\omega \wedge \ldots \wedge \omega}_{(n-1)\text{ times}} = \underbrace{\omega \wedge \ldots \wedge \omega}_{n\text{ times}}$. □

Problem 2.

The 3-sphere

$$\mathsf{S}^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 2\}$$

may be written as the union of the two solid tori

$$U := \{(z,w) \in \mathsf{S}^3 \mid |z|^2 \geq 1\} = \{(z,w) \in \mathsf{S}^3 \mid |w|^2 \leq 1\} \cong \mathsf{S}^1 \times \mathsf{B}^2,$$
$$V := \{(z,w) \in \mathsf{S}^3 \mid |z|^2 \leq 1\} = \{(z,w) \in \mathsf{S}^3 \mid |w|^2 \geq 1\} \cong \mathsf{B}^2 \times \mathsf{S}^1,$$

glued along the common boundary

$$\partial U = \partial V = \{(z,w) \in \mathsf{S}^3 \mid |z|^2 = |w|^2 = 1\} \cong \mathsf{S}^1 \times \mathsf{S}^1.$$

Thus $X \cong \mathsf{S}^3$, whereby $\pi_1(X) \cong \pi_1(\mathsf{S}^3) \cong 1$, since any n-sphere with $n \geq 2$ is simply connected. □

Problem 3.

By the above, $\mathsf{H}_j(X) \cong \mathsf{H}_j(\mathsf{S}^3) \cong \begin{cases} \mathbb{Z} & j = 0, 3, \\ 0 & \text{else.} \end{cases}$ □

Problem 4.

Background. In this problem we prove a form of Whitney's embedding theorem.

Fix some $v \in \mathsf{S}^{n-1}$, and let $x, y \in M$ with $x \neq y$. Then $\pi_v(x) = \pi_v(y) \iff x - y = cv$ for some $c \in \mathbb{R} \iff (x-y)/\|x-y\| = v$. So we see that the restriction $\pi_v\big|_M$ is injective if and only if v is not in the image of the smooth map $f : (M \times M) \setminus \Delta_M \to \mathsf{S}^{n-1}$ given by $f(x,y) := (x-y)/\|x-y\|$, where $\Delta_M := \{(x,x) \in M \times M\}$. In other words, $\pi_v\big|_M$ is injective for all $v \in \mathsf{S}^{n-1} \setminus \mathsf{im}(f)$, so it remains to check that $\mathsf{im}(f)$ has measure 0. But this holds by a corollary of Sard since the dimension of the domain is strictly less than that of the codomain,

$$\mathsf{dim}_{\mathbb{R}}((M \times M) \setminus \Delta_M) = 2 \cdot \mathsf{dim}_{\mathbb{R}}(M) \leq 2\left(\frac{n}{2} - 1\right) = n - 2 < n - 1 = \mathsf{dim}_{\mathbb{R}}(\mathsf{S}^{n-1}).$$

□

Problem 5.

See problem 5 of 2011, Spring.

Problem 6.

See problem 7 of 2007, Fall.