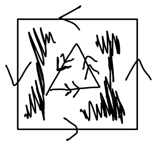
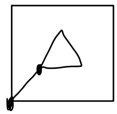
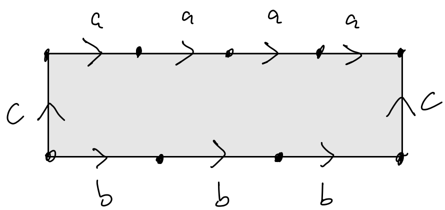
2016 Spring $\boxed{\#1}$

$X =$  Compute $\pi_1(X)$.

By making a cut from  the bottom left vertex of the square to the bottom left vertex of the triangle, and identifying the resulting edges, we get the diagram

$X =$ 

Hence $\pi_1(X) = \langle a, b, c : a^4 c^{-1} b^{-3} c = 1 \rangle$

2016 Spring $\boxed{\#2}$

X path connected $\pi_1(X; x_0) = \mathbb{Z}/5$

Covering space $\pi: \tilde{X} \to X$ 6-sheeted.

Claim: $\tilde{X}$ either has 2 or 6 connected components.

The connected covering spaces of X are classified by the subgps of $\pi_1(X; x_0) = \mathbb{Z}/5$. There are only two subgroups: the trivial subgp and the whole group, because $\langle 1 \rangle = \langle 2 \rangle = \langle 3 \rangle = \langle 4 \rangle = \mathbb{Z}/5$ since 5 prime.

Hence the trivial subgrp corresponds to a 5-sheeted covering space since the trivial subgp has index 5 in $\mathbb{Z}/5$. Similarly the subgroup $\mathbb{Z}/5 \subset \mathbb{Z}/5$ corresponds to a 1-sheeted covering space.

Therefore, a 6-sheeted covering space can only be made from six 1-sheeted covering spaces or one 5-sheeted and one 1-sheeted covering spaces.

i.e. $\tilde{X}$ either has 2 or 6 connected components.

Spring 2016 $\boxed{\#3}$

$X = S^1 \times S^n$, $n \geq 1$ compute $H_k(S^1 \times S^n; \mathbb{Z})$.

Let $A = \left(-\frac{3\pi}{4}, \frac{3\pi}{4}\right) \times S^n$, $B = \left(\pi/4, \frac{7\pi}{4}\right) \times S^n$.

Then $A \cap B \simeq S^n \sqcup S^n$, $A, B \simeq S^n$, $A \cup B = X$.

By M.V. we have

$$\begin{tikzcd}
\cdots \arrow[r] & H_k(A \cap B) \arrow[r] \arrow[d, phantom, "\wr"] & H_k(A) \oplus H_k(B) \arrow[r] \arrow[d, phantom, "\wr"] & H_k(X) \arrow[r] & \cdots \\
& H_k(S^n) \oplus H_k(S^n) & H_k(S^n) \oplus H_k(S^n) & &
\end{tikzcd}$$

For $\quad H_n(S^n) \oplus H_n(S^n) \longrightarrow H_n(S^n) \oplus H_n(S^n)$ we have

$$\mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$$
$$(a, b) \longmapsto (a+b, a+b)$$
$$\ker = \{(a, -a)\} \approx \mathbb{Z}, \quad \operatorname{im} = \{(a+b, a+b)\} \approx \mathbb{Z}$$

Then

$$\begin{array}{ccccccc}
0 \longrightarrow & H_{n+1}(X) & \longrightarrow & H_n(S^n) \oplus H_n(S^n) & \longrightarrow & H_n(S^n) \oplus H_n(S^n) \\
& & & \mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z}
\end{array}$$

$\Rightarrow H_{n+1}(X) \approx \mathbb{Z}$, the kernel of

and for $n \geq 2$,

$$\begin{array}{ccccccc}
H_n(S^n) \oplus H_n(S^n) & \longrightarrow & H_n(S^n) \oplus H_n(S^n) & \longrightarrow & H_n(X) & \longrightarrow & 0 \\
\mathbb{Z} \oplus \mathbb{Z} & & \mathbb{Z} \oplus \mathbb{Z} & & & &
\end{array}$$

which gives $H_n(X) \approx \operatorname{im}\left(H_n(S^n) \oplus H_n(S^n) \to H_n(X) \right) \approx \mathbb{Z}$

since the kernel of that map is $\mathbb{Z}$.

Sticking with $n \geq 2$, we have

$$0 \to H_1(X) \to H_0(S^n) \oplus H_0(S^n) \to H_0(S^n) \oplus H_0(S^n)$$
$$\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$$
$$(u, v) \longmapsto (u+v, u+v)$$

So again $H_1(X) \approx \mathbb{Z}$, and $H_0(X) \approx \mathbb{Z}$ b/c X path connected.

In sum,

For $n \geq 2$

$$\boxed{H_k(S^1 \times S^n) = \begin{cases} \mathbb{Z} & k = 0, 1, n, n+1 \\ 0 & \text{else} \end{cases}}$$

and

$$\boxed{H_k(S^1 \times S^1) = H_k(T^2) = \begin{cases} \mathbb{Z} & k = 0, 2 \\ \mathbb{Z} \oplus \mathbb{Z} & k = 1 \\ 0 & \text{else.} \end{cases}}$$

Spring 2016 $\boxed{\#4}$

M cpct oriented n-dim'l mfd.

$f: M \to \mathbb{R}^n$ differentiable, $f(M)$ non-empty interior in $\mathbb{R}^n$.

(a) Claim: $\exists\, x \in M$ s.t. $\exists$ nbd U of x s.t.

$f|_U : U \to f(U)$ is a diffeomorphism.

Let V be an open n-dim'l ball contained in $f(M)$.
Let $y \in f(M)$. Consider $f^{-1}(y) \subset M$. If df_x not surjective for any $x \in f^{-1}(y) \subset M$, then y is a critical value of f. By Sards theorem, the set of such y have measure 0 in $\mathbb{R}^n$. Then there must be some $y \in V$ s.t. $\exists\, x \in f^{-1}(y)$ with df_x surjective.
By dimension df_x is an isomorphism. Let W be a chart containing x. Then, df_x can be viewed as a matrix, and since det is a continuous function we can find U with $x \in U \subset W$ s.t.
$df_{\tilde{x}}$ isomorphic $\forall\, \tilde{x} \in U$, i.e. a diffeomorphism.

(b) Claim: $\exists$ at least two points $x, y \in M$ s.t. f is a local diffeo at x and y, f orientation preserving at x, and reversing at y.

Let $\omega \in \Omega^n(\mathbb{R}^n)$ s.t. ω vanishes on $f(M)$, is compactly supported and has $\int_{\mathbb{R}^n} \omega > 0$. Then

$$\deg(f)\int_{\mathbb{R}^n} w = \int_M f^* w = \int_M 0 = 0 \Rightarrow \deg(f) = 0.$$

Since degree of $f = 0$, then

for some regulr point $y \in f(M)$,

then $0 = \sum_{x \in f^{-1}(y)} \operatorname{sgn}(x) \Rightarrow$ thee must

be some x with $\operatorname{sgn}(x) = +1$ and

some with $\operatorname{sgn}(x) = -1$ in $f^{-1}(y)$.

This is to say (b).

Spring 2016 $\boxed{\#5}$

Is there an $\omega \in \Omega^n(\mathbb{RP}^n)$ s.t. $\omega(y) \neq 0$ at every $y \in \mathbb{RP}^n$?

$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k=0 \text{ or } k=n \text{ odd} \\ \mathbb{Z}/2 & k \text{ odd}, \quad 0<k<n \\ 0 & \text{else} \end{cases}$$

$\Rightarrow$ $\mathbb{RP}^n$ orientable for n odd

Since $H_n(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & n \text{ odd} \\ 0 & \text{else} \end{cases}$

? Cellular homology

2016 Spring $\boxed{\#6}$

$$H^k(S^n) \approx \begin{cases} 0 & k \neq 0, n \\ \mathbb{R} & k = 0, n \end{cases}$$

$f: S^{2n-1} \to S^n$ with $n \geq 2$. If $\alpha \in \Omega^n(S^n)$

with $\int_{S^n} \alpha = 1$, then

(a) $\exists \; \beta \in \Omega^{n-1}(S^{2n-1})$ s.t. $f^*(\alpha) = d\beta$.

(b) $I(f) := \int_{S^{2n-1}} \beta \wedge d\beta$ independent of α, β.

(a) α closed since $d\alpha \in \Omega^{n+1}(S^n)$ all zero.

$f^*(\alpha)$ closed since $d(f^*\alpha) = f^*(d\alpha) = 0$.

Then $[f^*(\alpha)] \in H^n(S^{2n-1}) = 0 \Rightarrow f^*(\alpha)$ exact.

(b) This is equivalent to the claim that

$[\beta \wedge d\beta] = [\beta' \wedge d\beta']$ for any α', β' satisfying

the same conditions.

$$f_*([\beta \wedge d\beta]) = 0 \in H^{2n-1}(S^n) = 0$$

?

Let α', β' be other forms satisfying the above conditions. Then WTS

$$\int_{S^{2n-1}} \beta \wedge d\beta = \int_{S^{2n-1}} \beta' \wedge d\beta'$$

$$d\beta' = f^*\alpha' , \quad \int_{S^n} \alpha' = \int_{S^n} \alpha = 1$$

$$\int_{S^{2n-1}} \beta \wedge d\beta = \int_{S^{2n}} \beta' \wedge d\beta$$

$$\int_{S^{2n-1}} \beta \wedge d\beta - \beta' \wedge d\beta = 0$$

i.e. $\beta \wedge d\beta - \beta' \wedge d\beta$ exact

$(\beta - \beta') \wedge d\beta$ exact?

$\beta - \beta'$ closed $\Rightarrow$ exact $\Rightarrow$

$\beta - \beta' = d\gamma \qquad \beta = d\gamma - \beta'$

$$d\gamma \wedge d\beta$$

$$\int_{S^{2n-1}} \beta \wedge d\beta = \int_{S^{2n-1}} (d\gamma - \beta) \wedge d\beta$$

$$= \int d\gamma \wedge d\beta +$$

$$\int_{S^{2n-1}} d(\gamma \wedge d\beta) = \int_{S^{2n-1}} d(\gamma \wedge d\beta) =$$

$$= \int_{B^{2n}} d^2(\gamma \wedge d\beta) =$$

α, α' $\quad \int_{S^n} \alpha = \int_{S^n} \alpha' = 1.$

$\exists\ \beta, \beta'$ with $d\beta, d\beta' = f^*(\alpha), f^*(\alpha')$

NTS $\displaystyle\int_{S^{2n-1}} \beta \wedge d\beta \stackrel{?}{=} \int_{S^{2n-1}} \beta' \wedge d\beta'$

$$\int_{S^{2n-1}} \beta \wedge f^*(\alpha) \stackrel{?}{=} \int_{S^{2n-1}} \beta' \wedge f^*(\alpha')$$

Stokes $=$ $\int_{B^{2n}} d\beta \wedge f^*(\alpha) \overset{?}{=} \int_{B^{2n}} d\beta' \wedge f^*(\alpha')$

$= \int_{B^{2n}}$