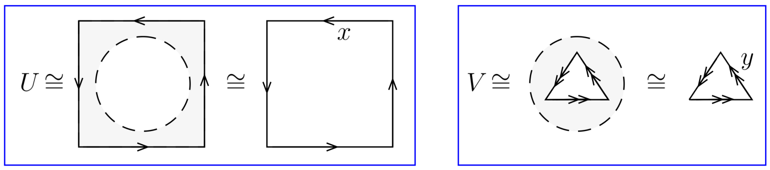
2016, Spring

Problem 1.

We write X as the union of the subspaces U and V shown below, with $U \cap V \cong \mathsf{S}^1$.



Let $x \in \pi_1(U)$ and $y \in \pi_1(V)$ correspond to the edges above as labeled. Letting $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ be the canonical inclusions, then the induced homomorphism $i_* : \pi_1(U \cap V) \to \pi_1(U)$ maps the single generator $1 \in \pi_1(U \cap V) \cong \pi_1(\mathsf{S}^1) \cong \mathbb{Z}$ to $i_*(1) = x^4$, and $j_* : \pi_1(U \cap V) \to \pi_1(V)$ maps it to $j_*(1) = y^3$. So by van Kampen,

$$\pi_1(X) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \cong \frac{\langle x, y \rangle}{\langle x^4 y^{-3} \rangle} = \big\langle x, y \mid x^4 = y^3 = 1 \big\rangle.$$

□

Problem 2.

- Letting $\mathsf{Bij}(p^{-1}(x_0))$ denote the set of bijections $p^{-1}(x_0) \to p^{-1}(x_0)$, we have an assignment

  $$F : \pi_1(X, x_0) \to \mathsf{Bij}(p^{-1}(x_0)), \quad F_{[\gamma]}(\tilde{x}) := \tilde{\gamma}_{\tilde{x}}(1),$$

  where $\tilde{\gamma}_{\tilde{x}} : [0,1] \to \tilde{X}$ is the unique lift of γ satisfying $\tilde{\gamma}_{\tilde{x}}(0) = \tilde{x}$. This assignment is precisely the monodromy action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$, and as such, for any $[\gamma] \in \pi_1(X, x_0)$, the order of $F_{[\gamma]}$ must divide $|\pi_1(X, x_0)| = |\mathbb{Z}_5| = 5$.

- Take some connected component $\tilde{X}_1 \subset \tilde{X}$, and a point $\tilde{x} \in \tilde{X}_1$. There exists a path in X from $\pi(\tilde{x})$ to x_0, and this path lifts to a path in $\tilde{X}_1$ from $\tilde{x}_0$ to an element of $p^{-1}(x_0)$. Since $\tilde{X}_1$ is connected, this means that this element of $p^{-1}(x_0)$ belongs to $\tilde{X}_1$.

  Next suppose a connected component $\tilde{X}_1 \subset \tilde{X}$ contains a distinct pair $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$. There exists a path $\tilde{\gamma}$ in $\tilde{X}_1$ from $\tilde{x}_0$ to $\tilde{x}_1$. Then $\pi \circ \tilde{\gamma}$ is a loop in X based at x_0, and $F_{[\pi \circ \tilde{\gamma}]}$ is nontrivial (it for instance sends $\tilde{x}_0$ to $\tilde{x}_1$).

  Thus if $F_{[\gamma]}$ has order 1 for each $[\gamma] \in \pi_1(X, x_0)$ (i.e. if the action F is trivial), then no connected component of $\tilde{X}$ contains more than one element of $p^{-1}(x_0)$. But we also showed that each connected component of $\tilde{X}$ contains at least one such element, and we conclude that $\tilde{X}$ has exactly one connected component for each element of $p^{-1}(x_0)$. So $\tilde{X}$ has six connected components.

- Finally, if F is nontrivial, then there's some $[\gamma] \in \pi_1(X, x_0)$ such that $F_{[\gamma]}$ has order 5, and as such, there's some connected component $\tilde{X}_1$ of $\tilde{X}$ containing at least 5 elements of $p^{-1}(x_0)$. The elements of $p^{-1}(x_0)$ contained in $\tilde{X}_1$ belong to a single orbit of the action F since $\tilde{X}_1$ is connected, and the order of any such orbit must divide $|\pi_1(X, x_0)| = 5$. Hence there are exactly 5 elements of $p^{-1}(x_0)$ contained in $\tilde{X}_1$, and the last element belongs to some other connected component of $\tilde{X}$. So $\tilde{X}$ has two connected components.

□

Problem 3.

By problem 5 of 2005, Fall, we have $\mathsf{H}_j(\mathsf{S}^n \times \mathsf{S}^1) \cong \mathsf{H}_j(\mathsf{S}^n) \oplus \mathsf{H}_{j-1}(\mathsf{S}^n)$ for all $j \in \mathbb{Z}$. Thus

$$\mathsf{H}_j(\mathsf{S}^1 \times \mathsf{S}^1) \cong \left(\begin{cases} \mathbb{Z} & j = 0, 1, \\ 0 & \text{else} \end{cases}\right) \oplus \left(\begin{cases} \mathbb{Z} & j = 1, 2, \\ 0 & \text{else} \end{cases}\right) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, \\ \mathbb{Z} & j = 2, \\ 0 & \text{else}, \end{cases}$$

and for $n \geq 2$,

$$\mathsf{H}_j(\mathsf{S}^n \times \mathsf{S}^1) \cong \left(\begin{cases} \mathbb{Z} & j = 0, n, \\ 0 & \text{else} \end{cases}\right) \oplus \left(\begin{cases} \mathbb{Z} & j = 1, n+1, \\ 0 & \text{else} \end{cases}\right) \cong \begin{cases} \mathbb{Z} & j = 0, 1, n, n+1, \\ 0 & \text{else}. \end{cases}$$

□

Problem 4.

(a) Since $\mathsf{im}(f)$ has nonempty interior, it has positive Lebesgue measure, so by Sard we may choose a regular value $y \in \mathsf{im}(f)$ of f. Now take some $x \in f^{-1}(y)$. Then $\mathsf{d}f_x : \mathsf{T}_x M \to \mathsf{T}_y \mathbb{R}^n$ is a surjective linear map of n-dimensional vector spaces, and thereby a linear isomorphism. So by the inverse function theorem, there exists an open neighborhood $U \subset M$ of x such that $f\big|_U : U \to f(U)$ is a diffeomorphism. □

(b) Since M is compact and f is continuous, $\mathsf{im}(f)$ is compact, and in particular not all of $\mathbb{R}^n$. So f isn't surjective, and $\mathsf{deg}(f) = 0$. Let $y \in \mathsf{im}(f)$ be as in part (a). Then

$$0 = \mathsf{deg}(f) = \sum_{x \in f^{-1}(y)} \mathsf{deg}_x(f).$$

Recall that $f^{-1}(y) \neq \varnothing$, and each local degree $\mathsf{deg}_x(f) = \pm 1$. So to get zero on the left-hand side, there must be points $x_1, x_2 \in f^{-1}(y)$ so that $\mathsf{deg}_{x_1}(f) = 1$ and $\mathsf{deg}_{x_2}(f) = -1$. Then f is orientation-preserving at x_1 and orientation-reversing at x_2. □

Problem 5.

Since $\mathbb{R}\mathsf{P}^n$ is an n-manifold, then there exists a nowhere vanishing volume form $\omega \in \Omega^n(\mathbb{R}\mathsf{P}^n)$ if and only if $\mathbb{R}\mathsf{P}^n$ is orientable. And by problem 3 of 2011, Spring, $\mathbb{R}\mathsf{P}^n$ is orientable if and only if either $n \geq 0$ is odd or (trivially) $n = 0$. □

Problem 6.

See problem 4 of 2008, Spring.