# Geometry and Topology Graduate Exam

Spring 2014

*Solve all* **SEVEN** *problems. Partial credit will be given to partial solutions.*

Sure. I will.

lifting ⇒ homeomorphic connectedness + compactness

**Problem 1.** Let $X_n$ denotes the complement of $n$ distinct points in the plane $\mathbb{R}^2$. Does there exist a covering map $X_2 \to X_1$? Explain.

homotope into good pairs

**Problem 2.** Let $D = \{z \in \mathbb{C}; |z| \leq 1\}$ denote the unit disk, and choose a base point $z_0$ in the boundary $S^1 = \partial D = \{z \in \mathbb{C}; |z| = 1\}$. Let $X$ be the space obtained from the union of $D$ and $S^1 \times S^1$ by gluing each $z \in S^1 \subset D$ to the point $(z, z_0) \in S^1 \times S^1$. Compute all homology groups $H_k(X; \mathbb{Z})$.

use induced homology

**Problem 3.** Let $B^n = \{x \in \mathbb{R}^n; \|x\| \leq 1\}$ denote the $n$-dimensional closed unit ball, with boundary $S^{n-1} = \{x \in \mathbb{R}^n; \|x\| = 1\}$. Let $f\colon B^n \to \mathbb{R}^n$ be a continuous map such that $f(x) = x$ for every $x \in S^{n-1}$. Show that the origin $0$ is contained in the image $f(B^n)$. (Hint: otherwise, consider $S^{n-1} \to B^n \xrightarrow{f} \mathbb{R}^n - \{0\}$.) $\to S^{n-1}$

let $f_*: \frac{\partial}{\partial x} \mapsto X$, $\frac{\partial}{\partial y} \mapsto Y$

check Jacobian solve ODE's. + inverse fn thm.

**Problem 4.** Consider the following vector fields defined in $\mathbb{R}^2$:

$$\mathbf{X} = 2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \qquad \text{and} \qquad \mathbf{Y} = \frac{\partial}{\partial y}.$$

Determine whether or not there exists a (locally defined) coordinate system $(s, t)$ in a neighborhood of $(x, y) = (0, 1)$ such that

$$\mathbf{X} = \frac{\partial}{\partial s}, \qquad \text{and} \qquad \mathbf{Y} = \frac{\partial}{\partial t}.$$

induced atlas + check jacobian determinant

**Problem 5.** Let $M$ be a differentiable (not necessarily orientable) manifold. Show that its cotangent bundle

$$T^*M = \{(x, u); x \in M \text{ and } u\colon T_xM \to \mathbb{R} \text{ linear}\}$$

is a manifold, and is orientable.

stokes thm.

**Problem 6.** Calculate the integral $\int_{S^2} \omega$ where $S^2$ is the standard unit sphere in $\mathbb{R}^3$ and where $\omega$ is the restriction of the differential 2-form

$$(x^2 + y^2 + z^2)(x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy)$$

sard's theorem.

**Problem 7.** Let $M$ be a compact $m$-dimensional submanifold of $\mathbb{R}^m \times \mathbb{R}^n$. Show that the space of points $x \in \mathbb{R}^m$ such that $M \cap \mathbb{R}^n$ is infinite has measure $0$ in $\mathbb{R}^m$.

Spring 2014

① $X_n = \mathbb{R}^2 - \{n\text{-pts}\}$

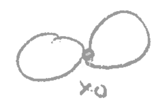
Does $\exists$ a covering map $X_2 \to X_1$? Explain.

Pf $\mathbb{R}^2 - \{pt\} = X_1 \simeq S^1$

$\mathbb{R}^2 - \{2pts\} = X_2 \simeq S^1 \vee S^1$

However $\pi_1(S^1) = \mathbb{Z}$ so if $p: S^1 \vee S^1 \to S^1$ is a cover

$\Rightarrow p_*(\pi_1(S^1 \vee S^1)) \leq \mathbb{Z}$, so $p_*(\pi_1(S^1 \vee S^1)) = m\mathbb{Z}$

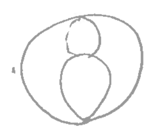
but $\pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$. Furthermore since $S^1 \vee S^1 \cong$ 

locally $\overset{x_0}{X}$ cannot be homeomorphic to any nbhd of $S^1$, so

$S^1 \vee S^1$ cannot be a cover of $S^1$.

② $D = \{|z| \leq 1\}$; $X$ be the union of $D$ & $S^1 \times S^1$ by

gluing each $z \in S^1 \subseteq D$ to $(z, z_0) \in S^1 \times S^1$. Find $H_k(X, \mathbb{Z})$.

Pf Notice that $X \cong$  $\simeq$  $\simeq S^2 \vee S^1$

so $\tilde{H}_k(S^2 \vee S^1) = \tilde{H}_k(S^2) \oplus \tilde{H}_k(S^1)$ since $(S^1, pt)$ & $(S^2, pt)$

are good pairs. So $H_k(X) = \begin{cases} \mathbb{Z} & k = 0, 1, 2 \\ 0 & \text{else.} \end{cases}$

③ $B^n = \{x \in \mathbb{R}^n, \|x\| \le 1\}$, let $f: B^n \to \mathbb{R}^n$ be continuous such that $f(x) = x$ $\forall x \in S^{n-1}$. Show $0 \in f(B^n)$.

Pf Suppose not. Then $S^{n-1} \xhookrightarrow{i} B^n \xrightarrow{f} \mathbb{R}^n - \{0\} \xrightarrow{r} S^{n-1}$

$x \longmapsto \frac{x}{|x|}$

then $r \circ f \circ i = id_{S^{n-1}} \Rightarrow (r \circ f \circ i)^* = id^*$

However $0 = H_n(S^{n-1}) \xrightarrow{i^*} H_n(B^n) \Rightarrow i^* \equiv 0$.

So $(r \circ f \circ i)^* = 0 \Rightarrow \Leftarrow$. Thus $0 \in f(B^n)$.

④ Consider $X = 2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ & $Y = \frac{\partial}{\partial y}$

and determine whether or not $\exists$ a locally defined coordinate system $(s,t)$ in a nbd of $(0,1)$ $\ni$ $X = \partial/\partial s$ & $Y = \frac{\partial}{\partial t}$.

Pf let $f(x,y) = (s,t)$ be such a parametrization.

So $s = f_1$ & $t = f_2$. Then we have maps:

$$\begin{array}{ccc} & & \mathbb{R}^2\ (x,y) \\ & & \downarrow id \\ \mathbb{R}^2 & \xrightarrow{f} & \mathbb{R}^2 \\ (x,y) & & \end{array}$$

that induce maps on the tangent spaces so that $f_*: T_p\mathbb{R}^2 \to T_{f(p)}\mathbb{R}^2$

$\Rightarrow f_*\left(\frac{\partial}{\partial x}\right) = X$

$f_*\left(\frac{\partial}{\partial y}\right) = Y$

where:

$$f_* = \begin{array}{cc} \begin{array}{cc} \partial/\partial x & \partial/\partial y \end{array} & \\ \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix} & \begin{array}{c} \partial/\partial x \\ \partial/\partial y \end{array} \end{array}$$

So then we obtain a system of diff'l eqn's.

$$f_*\left(\frac{\partial}{\partial x}\right) = \frac{\partial f_1}{\partial x}\cdot\frac{\partial}{\partial x} + \frac{\partial f_2}{\partial x}\cdot\frac{\partial}{\partial y} = X = 2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

$$\text{\& } f_*\left(\frac{\partial}{\partial y}\right) = \frac{\partial f_1}{\partial y}\cdot\frac{\partial}{\partial x} + \frac{\partial f_2}{\partial y}\cdot\frac{\partial}{\partial y} = Y = \frac{\partial}{\partial y}$$

$$\Rightarrow \quad \frac{\partial f_1}{\partial x} = 2 \qquad \frac{\partial f_2}{\partial x} = x \quad \Rightarrow f_1 = 2x + C(y) = 2x + C$$

$$\frac{\partial f_1}{\partial y} = 0 \qquad \frac{\partial f_2}{\partial y} = 1 \quad \Rightarrow f_2 = \frac{x^2}{2} + y + D$$

Hence if such a parametrization existed it would have to satisfy the equations above.

Hence around $(0,1)$ we see that if $0 = 2x + C \Rightarrow C = 0$, $x = 0$

$\&$ $1 = \frac{x^2}{2} + y + D = y + D \Rightarrow D = 0$ $y = 1$

Hence $\{f_1 = 2x, f_2 = \frac{x^2}{2} + y\}$ are candidates. To see if they form a local coordinate system we use the inverse function theorem:

$$\det(f_*) = \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_2}{\partial x}\cdot\frac{\partial f_1}{\partial y} = 2 \neq 0 \quad \text{so by IFT}$$

$f$ is a local diffeomorphism around $(0,1)$, $\&$ hence a local coordinate system.

⑥ Calculate : $\int_{S^2} \omega$ , $\omega = (x^2+y^2+z^2)(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy)$

Pf $\int_{S^2} \omega = \int_{S^2} x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$

by stokes $= \int_{B^2} 3\,dx\wedge dy\wedge dz = 3 \cdot vol(B^2)$

⑦ M compact m-dim submanifold of $\mathbb{R}^m \times \mathbb{R}^n$.

Show the space of points $x \in \mathbb{R}^m \ni M \cap \mathbb{R}^n$ is infinite has measure 0 in $\mathbb{R}^m$.

Pf Consider

$$\begin{array}{ccc} M & \overset{i}{\hookrightarrow} & \mathbb{R}^m \times \mathbb{R}^n \\ & {\scriptstyle f = \pi \circ i} \searrow & \downarrow \pi \\ & & \mathbb{R}^m \end{array}$$

Then $f = \pi \circ i$ is smooth so the set of critical values has measure zero in $\mathbb{R}^m$. So then let $x \in \mathbb{R}^m \cap M \ni x$ is a regular value.

Then $f_{*x}: T_p M \to T_q \mathbb{R}^m$ is surjective & nonzero, so locally by the inverse function theorem, $x$ is locally diffeomorphic to $f^{-1}(x)$.

so then if $y \in f^{-1}(x)$ $\exists$ $U_y \ni y \in U_y \simeq f(U) = W \ni x$

Now, $\{x\}$ is closed in $\mathbb{R}^m$, so $f^{-1}(x)$ is closed in $M$ & hence compact since $M$ is compact. However, $\bigcup_{y \in f^{-1}(x)} U_y$ is an open cover for $f^{-1}(x)$ so by compactness, $\exists$ finite subcover, so $f^{-1}(x) \subseteq \bigcup_{i=1}^{k} U_{y_i}$

Now since each $\mathcal{U}_{y_i} \simeq W$, if $\mathcal{U}_{y_\alpha} \cap \mathcal{U}_{y_\beta} \neq \emptyset \Rightarrow \mathcal{U}_{y_\alpha} = \mathcal{U}_{y_\beta}$

Hence $\exists$ finitely many points in the fiber of $x$.

Thus $\{y : (x,y) \in M\}$ is finite.

In particular this implies the fibers of all regular points are finite so necessarily, all points w/ infinite fibers are critical, which by Sard's theorem, have measure zero.