2014 Spring $\boxed{\#1}$

2014 Spring $\boxed{\#2}$

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2014 Spring $\boxed{\#4}$

Consider the following vect. Fields in $\mathbb{R}^2$:

?

$$X = 2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad Y = \frac{\partial}{\partial y}.$$

Determine whether or not $\exists$ (locally defined) coordinate system (s,t) in a nbd of $(x,y)=(0,1)$ s.t.

$$X = \frac{\partial}{\partial s}, \quad Y = \frac{\partial}{\partial t}$$

Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $\phi(x,y) = (s(x,y), t(x,y))$

Then $\frac{\partial}{\partial s} = d\phi_p\left(\frac{\partial}{\partial x}\right)$ and $\frac{\partial}{\partial t} = d\phi_p\left(\frac{\partial}{\partial y}\right)$

$$= \frac{\partial s}{\partial x}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x}\frac{\partial}{\partial y} \qquad = \frac{\partial s}{\partial y}\frac{\partial}{\partial x} + \frac{\partial t}{\partial y}\frac{\partial}{\partial y}$$

Then we want

$$\frac{\partial s}{\partial x} = 2, \quad \frac{\partial t}{\partial x} = x, \qquad \frac{\partial s}{\partial y} = 0, \quad \frac{\partial t}{\partial y} = 1$$

This is satisfied by $s = 2x$, $t = \frac{1}{2}x^2 + y$ e.g.

Then at $(0,1)$, $d\phi_{(0,1)} = \begin{pmatrix} 2 & 0 \\ x & 1 \end{pmatrix}\Big|_{(0,1)} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ invertible.

$\Rightarrow$ by inverse function thm. $\exists$ nbd of $(0,1)$

s.t. ϕ is a diffeomorphism.

2014 Spring $\boxed{\#5}$

M diff. mfd. $T^*M = \{(x,u) ; x \in M, u : T_xM \to \mathbb{R} \text{ linear}\}$

Claim: T^*M orientable mfd.

Let (V,ϕ) be a chart in M, $\phi : V \xrightarrow{\cong} \phi(V) \subset \mathbb{R}^n$.

Let $\tilde{V} = \{(x,u) \in T^*M : x \in V\}$ and $\tilde{\phi} : \tilde{V} \to \mathbb{R}^n \times \mathbb{R}^n$

be defined by $\tilde{\phi}(x,u) = \left(\phi(x), u\left(\frac{\partial}{\partial x^1}\right), \dots, u\left(\frac{\partial}{\partial x^n}\right)\right)$.

Which is a homeomorphism since ϕ is a homeomorphism and u is linear. Thus we see that T^*M is a mfd.

An orientation form for T^*M would be a $2n$-form ω, nonvanishing.

Define $\omega_{(x,u)} : \left(T_{(x,u)}(T^*M)\right)^{2n} \longrightarrow \mathbb{R}$

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2014 Spring $\boxed{\#7}$

M compact m-dim'l submfd of $\mathbb{R}^m \times \mathbb{R}^n$.
Let M_x for $x \in \mathbb{R}^m$ be the translation of M by x in $\mathbb{R}^m \times \mathbb{R}^n$. Then the space of points $x \in \mathbb{R}^m$ s.t. $M_x \cap \mathbb{R}^n$ infinite has measure 0 in $\mathbb{R}^m$.

Define $f: M \subset \mathbb{R}^m \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ Then f sends

$$(x,y) \longmapsto x$$

a point in M to the $x \in \mathbb{R}^m$ s.t. $M_x \cap \mathbb{R}^n$ contains that point. Assume f sends an infinite amount of points in M to some x_0. (i.e. $M_{x_0} \cap \mathbb{R}^n$ infinite). We claim x_0 is a critical value. If not, then by regular level set Thm, $f^{-1}(x_0)$ is a submfd of M of dimension $m - m = 0$. I.e. $f^{-1}(x_0)$ is a collection of distinct points in M. Since f continuous and $\{x_0\}$ closed then $f^{-1}(x_0)$ closed and therefore compact b/c M compact. By the subspace topology the points in $f^{-1}(x_0)$ are open, so there can only be finitely many, otherwise $\{\{y\} : y \in f^{-1}(x_0)\}$ is an infinite open cover with no finite subcover. Hence x_0 critical value and by Sard's they have measure 0 in $\mathbb{R}^m$.