

Geometry Spring 2014 (Incomplete)

① X_n is the complement of n distinct points in the plane $\mathbb{R}^2$. Is there a covering map $X_2 \rightarrow X_1$?

solution

$X_1 \xrightarrow{p_n} X_1$ is a covering map,

where $p_n(z) = z^n$. $\pi_1(X_1; x_0) \cong \mathbb{Z}$.

The image of $(p_n)_* : \mathbb{Z} \rightarrow \mathbb{Z}$ is $n\mathbb{Z}$.

Since X_1 is path connected, locally path connected, and semilocally simply connected, every path

connected covering X of X_1 is determined by the image of $\pi_1(X; \tilde{x}_0)$ in $\pi_1(X_1; x_0)$ (even w/out base points, since $\pi_1(X_1; x_0)$ is abelian). We've constructed covering spaces for every subgroup $n\mathbb{Z}$ of $\mathbb{Z}$, so we know therefore that every covering space of X_1 is either simply connected or homeomorphic to X_1 . Since X_1 is not

homeomorphic to X_2 (the latter's fundamental group is the free group on two generators) and X_2 is not simply connected, it cannot be a covering space of X_1 (nor can $X_n, n > 1$).
 (Note in fact the universal cover of $\mathbb{R}^2 - \{0\}$ is $\mathbb{R}^2$) $\square$.

② $D = \{ z \in \mathbb{C} : |z| \leq 1 \}$ and choose a

base point z_0 in the boundary $S^1 = \partial D = \{ z \in \mathbb{C} : |z| = 1 \}$.

Let X be the space obtained as $(D \sqcup (S^1 \times S^1)) / \sim$

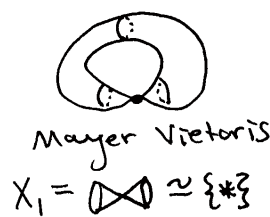
where $\sim$ is determined by the gluing map

$S^1 = \partial D \xrightarrow{\varphi} S^1 \times S^1$ where $\varphi(z) = (z, z_0)$. Compute the homology groups $H_k(X; \mathbb{Z})$.

solution



$\sim$
homotopy
equivalent



$$X_1 = \text{torus} \simeq \{*\}$$

$$\begin{array}{c} X_2 \\ \parallel \\ \square \\ \parallel \\ \mathbb{I} \times S^1 \\ \parallel \\ \mathbb{R} S^1 \end{array}$$

$$\dots \rightarrow H_k(S' \amalg S') \rightarrow H_k(\{*\}) \oplus H_k(S') \rightarrow H_k(X) \rightarrow H_{k-1}(S' \amalg S') \rightarrow$$

$$k > 2 / \rightarrow 0 \rightarrow 0 \rightarrow H_k(X) \rightarrow 0 \rightarrow$$

$$\boxed{H_k(X) \cong 0}$$

$$k=2 / \rightarrow 0 \rightarrow 0 \rightarrow H_2(X) \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{g} \mathbb{Z} \rightarrow$$

$(x, y) \mapsto x+y$

$$H_2(X) \cong \operatorname{im} f \cong \ker g = \{(x, y) : x+y=0\} = \{(x, -x)\} \cong \mathbb{Z}.$$

$$\boxed{H_2(X) \cong \mathbb{Z}}$$

$$k=1 / \begin{array}{c} (x, y) \mapsto x+y \\ \downarrow f \\ \mathbb{Z} \oplus \mathbb{Z} \end{array} \xrightarrow{g} \mathbb{Z} \xrightarrow{\cong} H_1(X) \xrightarrow{h} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\ell} \mathbb{Z} \oplus \mathbb{Z} \rightarrow$$

$(x, y) \mapsto (-x-y, x+y)$

$$\boxed{H_1(X) \cong \mathbb{Z}}$$

$$\boxed{H_0(X) \cong \mathbb{Z}}$$

$$\ker g \cong \operatorname{im} f = \mathbb{Z} \Rightarrow g=0.$$

$$H_1(X) \cong \operatorname{im} h \cong \ker \ell = \{(x, y) : x+y=0\} = \{(x, -x)\} \cong \mathbb{Z}.$$

□.

③ Let $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, let

$S^{n-1} = \partial B^n = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Let $f: B \rightarrow \mathbb{R}^n$

be continuous s.t. $f(x) = x$ for all $x \in S^{n-1}$

Show the origin is contained in the image $f(B^n)$.
We Assume $n \geq 2$.

Solution

Suppose otherwise.

Consider
$$S^{n-1} \xrightarrow{i} B^n \xrightarrow{f} \mathbb{R}^n - \{0\}$$

$$\downarrow \pi$$
$$S^{n-1}$$

So this induces a map

$$H_{n-1}(S^{n-1}) \rightarrow H_{n-1}(B^n) \rightarrow H_{n-1}(S^{n-1})$$


However, $\pi \circ f \circ i$ equals the identity $S^{n-1} \rightarrow S^{n-1}$,

so the map $H_{n-1}(S^{n-1}) \rightarrow H_{n-1}(S^{n-1})$ is also the identity map, in addition to being the zero map.

This contradicts $H_{n-1}(S^{n-1}) \cong \mathbb{Z}$.

□

④

vector fields in $\mathbb{R}^2$:

$$X = 2\partial_x + x\partial_y,$$

$$Y = \partial_y.$$

Determine whether there is a local coordinate system (s, t) around $(x, y) = (0, 1)$ s.t. $X = \partial_s, Y = \partial_t$.

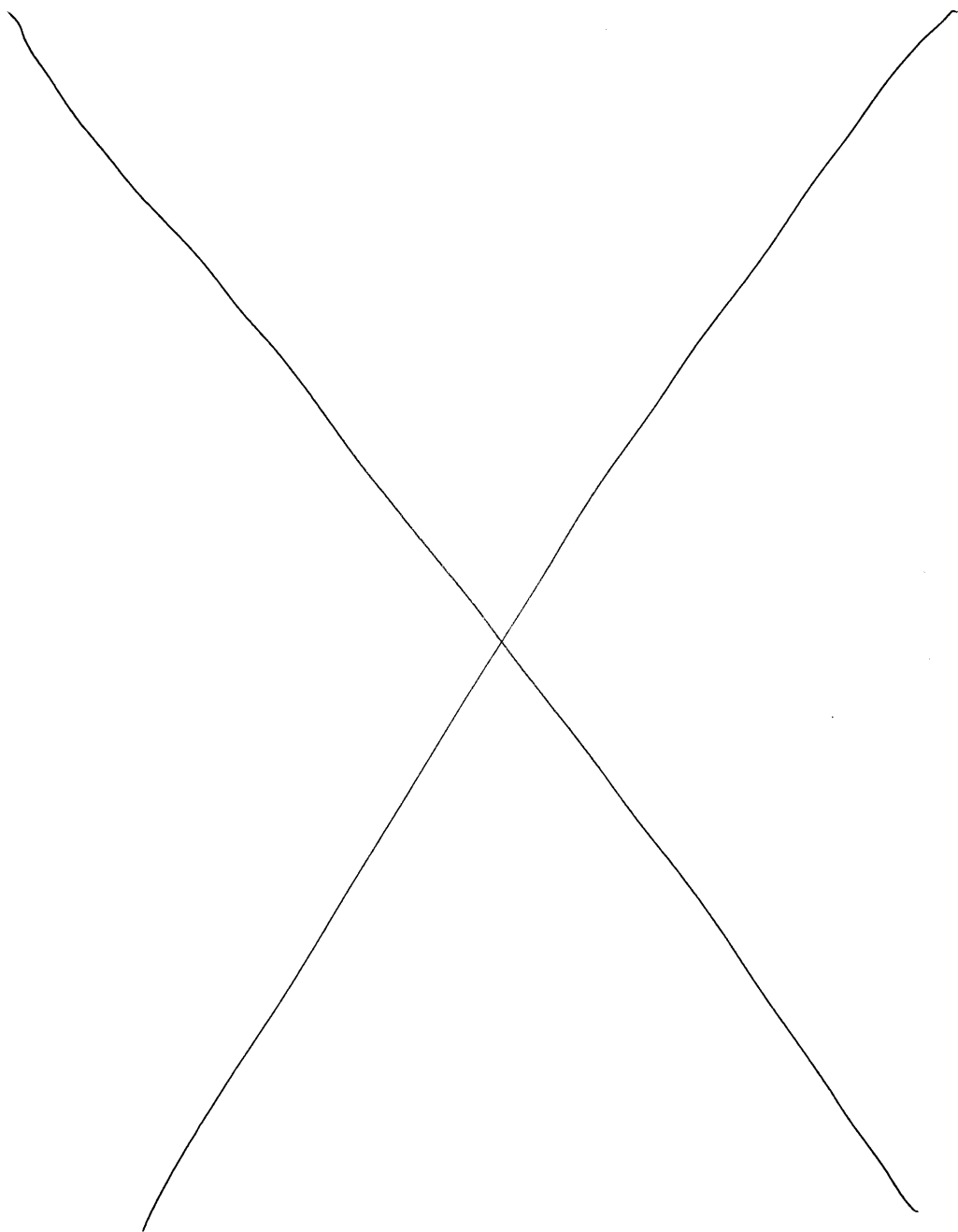
Solution

It is a theorem that
(from Bonahon's HW)
if $[X, Y] = 0$, then there is
such a local coordinate system around

every point. Recall $[\partial_i, \partial_j] = 0$ and $[fX, gY] = f(Xg)Y - g(Yf)X + fg[X, Y]$.

$$[X, Y] = 2[\partial_x, \partial_y] + [x\partial_y, \partial_y] =$$

$$= 0 + x\partial_y(\partial_y) - \partial_y(x)\partial_y + x[\partial_y, \partial_y] = 0. \quad \square$$



⑤ (augmented) M smooth mfd, Not necessarily oriented. (a) Show the tangent bundle TM is a manifold that is orientable.

Solution to (a) Let $\{U_i, \varphi_i\}_{i \in I}$ be an atlas for M .

$$TM = \{ (x, v) : x \in M, v \in T_x M \}$$

For an atlas take $\{\tilde{U}_i, \tilde{\varphi}_i\}_{i \in I}$, where

$$\tilde{U}_i := \pi^{-1}(U_i) \quad (\text{here } \pi: TM \rightarrow M \text{ is}$$

the projection $\pi(x, v) = x$) and

$$\tilde{\varphi}_i: \tilde{U}_i \rightarrow \mathbb{R}^{2m} \text{ is defined by}$$

$$\tilde{\varphi}_i(x, v) := (\varphi_i(x), T_x \varphi_i(v)).$$

Let $V_i := \varphi_i(U_i)$. The image of $\tilde{\varphi}_i$ is $V_i \times \mathbb{R}^{2m}$, and $\tilde{\varphi}_i$ has inverse given

by $\tilde{\varphi}_i^{-1} : V_i \times \mathbb{R}^m \rightarrow TM$ defined by

$$\tilde{\varphi}_i^{-1}(y, w) = (\varphi_i^{-1}(y), (T_x \varphi_i)^{-1}(w)).$$

Since φ_i is a homeomorphism, φ_i^{-1} is continuous, so clearly $\tilde{\varphi}_i$ and $\tilde{\varphi}_i^{-1}$ are continuous; so $\tilde{\varphi}_i$ is a homeomorphism. ✓

Clearly $\bigcup_{i \in I} \tilde{U}_i = TM$ since

$$\bigcup_{i \in I} U_i = M. \quad \checkmark$$

Let $\tilde{\varphi}_i$ and $\tilde{\varphi}_j$ be given. we NTS

$$\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1} : \tilde{\varphi}_i(\tilde{U}_i \cap \tilde{U}_j) \longrightarrow \tilde{\varphi}_j(\tilde{U}_i \cap \tilde{U}_j)$$

$\parallel$ $\mathbb{R}^m$ $\parallel$
 $\varphi_i(U_i \cap U_j) \times \mathbb{R}^m$ $\varphi_j(U_i \cap U_j) \times \mathbb{R}^m$

smooth.

we have the explicit formula

$$\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1} : (y, w) \mapsto (\varphi_i^{-1}(y), T_y \varphi_i^{-1}(w)) \mapsto (\varphi_j \circ \varphi_i^{-1}(y), T_{\varphi_i^{-1}(y)} \varphi_j(T_y \varphi_i^{-1}(w))).$$

The first coordinate is smooth because M is a mfd.

The second coordinate is smooth because it is ^{essentially} linear. ✓

So TM is a smooth manifold w/ this atlas.

We want to show this atlas is orientable. That is, $T_{(y,w)}(\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1}) : \mathbb{R}^{2m} \rightarrow \mathbb{R}^{2m}$ has positive determinant. Again,

$\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1}$ is given by $(y,w) \mapsto (\varphi_j \circ \varphi_i^{-1}(y), T_y(\varphi_j \circ \varphi_i^{-1})(w))$.

$$T_{(y,w)}(\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1})(A, B) = \frac{d}{dt} \bigg|_{t=0} \left(\varphi_j \circ \varphi_i^{-1}(\alpha(t)), T_{\alpha(t)}(\varphi_j \circ \varphi_i^{-1})(\omega + tB) \right)$$

where α is a path in M s.t. $\alpha(0) = y$ and $\alpha'(0) = A$

because

$$T_{(y,w)}(\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1})(A, w) = \frac{d}{dt} \bigg|_{t=0} (\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1})(\alpha(t)) \quad \text{---}$$

where $\alpha(t) := (\alpha(t), w + tB)$ is a path in

$$\varphi_i(U_i \cap U_j) \times \mathbb{R}^m \quad w/ \quad \alpha(0) = (y, w) \text{ and } \alpha'(0) = (A, B) \quad \text{---} \quad =$$

$$= \left. \frac{d}{dt} \right|_{t=0} \left(\varphi_j \circ \bar{\varphi}_i^{-1}(\alpha(t)), T_{\alpha(t)}(\varphi_j \circ \bar{\varphi}_i^{-1})(\omega + tB) \right).$$

$$\text{So } T_{(\gamma, \omega)}(\varphi_j \circ \bar{\varphi}_i^{-1})(A, B) = \left(T_{\gamma}(\varphi_j \circ \bar{\varphi}_i^{-1})(A), \left. \frac{d}{dt} \right|_{t=0} T_{\alpha(t)}(\varphi_j \circ \bar{\varphi}_i^{-1})(\omega + tB) \right).$$

(not to be confused w/ the i, j of $\varphi_j \circ \bar{\varphi}_i^{-1}$)

The (i, j) -component $(i, j = 1, \dots, m)$ of the Jacobian J is when $A = e_i$ and $B = 0$

~~$J_{ij} =$~~

The Jacobian is $J = \begin{pmatrix} | & | & | & | & | & | & | & | \\ | & | & | & | & | & | & | & | \\ \hline | & | & | & | & | & | & | & | \end{pmatrix}$.

(not to be confused w/ the j of $\varphi_j \circ \bar{\varphi}_i^{-1}$)

The j^{th} column $(j = 1, \dots, m)$ of the 1-1 quadrant equals

$$T_{\gamma}(\varphi_j \circ \bar{\varphi}_i^{-1})(e_j) = (j^{\text{th}} \text{ column of the Jacobian of } \varphi_j \circ \bar{\varphi}_i^{-1}).$$

The j^{th} column of the 1-2 quadrant equals $T_{\gamma}(\varphi_j \circ \bar{\varphi}_i^{-1})(0) = 0$.

The j^{th} column of the 2-2 quadrant is evaluated w/ $\alpha(t) = \gamma$ the constant path, as $A = 0$; so it just becomes

$$\left. \frac{d}{dt} \right|_{t=0} T_{\gamma}(\varphi_j \circ \bar{\varphi}_i^{-1})(\omega + te_j) = T_{\gamma}(\varphi_j \circ \bar{\varphi}_i^{-1})(e_j) =$$

$=$ (j^{th} column of the Jacobian of $\varphi_j \circ \tilde{\varphi}_i^{-1}$) . Therefore,

$$\text{Det } \mathcal{J} = \text{Det} \left(\underbrace{\text{Jacobian of } \varphi_j \circ \tilde{\varphi}_i^{-1}}_{\text{invertible}} \right)^2 > 0.$$

we conclude $\{ \tilde{u}_i, \tilde{\varphi}_i \}_{i \in I}$ is an orientable atlas for TM . $\square$.

(b) Show the cotangent bundle T^*M is an orientable manifold.

solution to (b). Again $\dim M = m$. $\{u_i, \varphi_i\}_{i \in I}$ is a chart.

$$T^*M = \{ (X, u) : X \in M \text{ and } u : T_X M \rightarrow \mathbb{R} \text{ linear} \}.$$

We define an atlas $\{ \tilde{u}_i, \tilde{\varphi}_i \}_{i \in I}$ on T^*M as follows: $\tilde{u}_i := \pi^{-1}(u_i)$ where $\pi : T^*M \rightarrow M$ is

defined by $\pi(X, u) = X$, and $\tilde{\varphi}_i : \tilde{u}_i \rightarrow \mathbb{R}^{2m}$ is defined

$$\text{by } \tilde{\varphi}_i(X, u) = (\varphi_i(X), (u(\partial^1_i(X)), \dots, u(\partial^m_i(X))))$$

where $\partial^j_i(X) := T_{\varphi_i(X)}(\tilde{\varphi}_i^{-1})(e_j)$. we construct the

inverse of $\tilde{\varphi}_i$. Since a linear form u is determined by its image on the basis $\{\partial_1^i(x), \dots, \partial_m^i(x)\}$, for any $w \in \mathbb{R}^m$ let u_w^i be the linear form $T_x M \rightarrow \mathbb{R}$ defined by $u_w^i(\partial_j^i(x)) = w_j$. Then the inverse $\tilde{\varphi}_i^{-1} : \tilde{\varphi}_i(\tilde{U}_i) \times \mathbb{R}^m \rightarrow T^*M$ of $\tilde{\varphi}_i$ is defined by $\tilde{\varphi}_i^{-1}(y, w) = (\tilde{\varphi}_i^{-1}(y), u_w^i)$. Clearly $\tilde{\varphi}_i$ and $\tilde{\varphi}_i^{-1}$ are continuous. $\checkmark$ (Note: The topology on T^*M has a basis given by $\{\tau^{-1}(U) : U \subset M \text{ open}\}.$)

Clearly $T^*M = \bigcup_{i \in I} \tilde{U}_i$ since $M = \bigcup_{i \in I} U_i$. $\checkmark$

Lastly we need $\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1} : \tilde{\varphi}_i(\tilde{U}_i \cap \tilde{U}_j) \times \mathbb{R}^m \rightarrow \tilde{\varphi}_j(\tilde{U}_i \cap \tilde{U}_j) \times \mathbb{R}^m$ to be smooth. The formula for $\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1}$ is given by $(y, w) \mapsto (\tilde{\varphi}_i^{-1}(y), u_w^i) \mapsto$

$(\varphi_j \circ \tilde{\varphi}_i^{-1}(y), (u_w^i(\partial_1^i(\tilde{\varphi}_i^{-1}(y))), \dots, u_w^i(\partial_m^i(\tilde{\varphi}_i^{-1}(y))))$ which is

smooth since $\varphi_j \circ \tilde{\varphi}_i^{-1}(y)$ is smooth and the second coordinate is essentially linear. $\checkmark$

We now want to show the Jacobian J of $\tilde{\varphi}_j \circ \tilde{\varphi}_i^{-1}$ has positive determinant. INCOMPLETE.