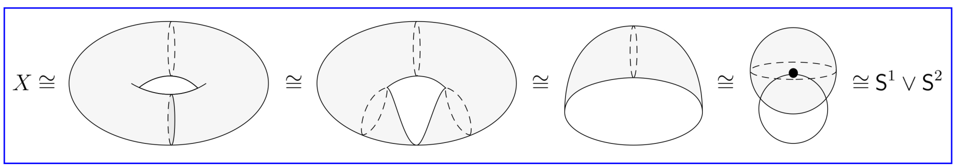
2014, Spring

Problem 1.

No. Similarly to problem 1 of 2005, Fall, we see that $X_1 \cong \mathsf{S}^1$ and $X_2 \cong \mathsf{S}^1 \vee \mathsf{S}^1$. Equivalence classes of connected covers of X_1 are in bijection with the subgroups of $\pi_1(X_1) \cong \pi_1(\mathsf{S}^1) \cong \mathbb{Z}$, and each such subgroup is of the form $k\mathbb{Z}$ for some $k \geq 0$. We know that the identity subgroup corresponds to the simply connected universal cover $\mathbb{R} \twoheadrightarrow \mathsf{S}^1$, and that for any $k \geq 1$, the subgroup $k\mathbb{Z}$ corresponds to the cover $\mathsf{S}^1 \twoheadrightarrow \mathsf{S}^1$ given by $z \mapsto e^{2\pi i/k} z$. Therefore if $X_2 \twoheadrightarrow X_1$ is indeed a (connected) cover, then by the above, X_2 is either simply connected or homeomorphic to S^1. But $\pi_1(X_2) \cong \pi_1(\mathsf{S}^1 \vee \mathsf{S}^1) \cong \mathsf{F}_2$ is nontrivial and not isomorphic to $\pi_1(\mathsf{S}^1) \cong \mathbb{Z}$, so neither of these is a possibility. □

Problem 2.

The space X is created by gluing each point $z \in \partial D$ to a corresponding point $(z, z_0) \in \mathsf{S}^1 \times \mathsf{S}^1$ on the meridianal circle on $\mathsf{S}^1 \times \mathsf{S}^1$ in which the second angular coordinate is fixed at z_0. Since D is contractible, we may shrink it to a point, thereby producing a "croissant." We then transform the shape until we're left with the wedge $\mathsf{S}^1 \vee \mathsf{S}^2$ shown below.



Then immediately $\mathsf{H}_j(X) \cong \mathsf{H}_j(\mathsf{S}^1 \vee \mathsf{S}^2) \cong \begin{cases} \mathbb{Z} & j = 0, 1, 2, \\ 0 & \text{else.} \end{cases}$ □

Problem 3.

See problem 4 of 2007, Fall.

Problem 4.

Yes. Assume that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a reparametrization $f(x, y) = (s, t)$, where $f_1(x, y) = s$ and $f_2(x, y) = t$ satisfy $X = \frac{\partial}{\partial s} = \mathsf{d}f(\frac{\partial}{\partial x})$ and $Y = \frac{\partial}{\partial t} = \mathsf{d}f(\frac{\partial}{\partial t})$ in some neighborhood of $(0, 1)$. Then

$$2\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = X = \mathsf{d}f\left(\frac{\partial}{\partial x}\right) = \frac{\partial f_1}{\partial x}\frac{\partial}{\partial x} + \frac{\partial f_2}{\partial x}\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial y} = Y = \mathsf{d}f\left(\frac{\partial}{\partial y}\right) = \frac{\partial f_1}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f_2}{\partial y}\frac{\partial}{\partial y}$$

and we have the system of equations

$$\frac{\partial f_1}{\partial x} = 2, \quad \frac{\partial f_2}{\partial x} = x, \quad \frac{\partial f_1}{\partial y} = 0, \quad \frac{\partial f_2}{\partial y} = 1.$$

Solving this yields $f_1(x, y) = 2x + c_1$ and $f_2(x, y) = \frac{1}{2}x^2 + y + c_2$ for some $c_1, c_2 \in \mathbb{R}$. If the (soon-to-be) local coordinate system given by f is centered at $(0, 1)$, then

$$(0, 0) = f(0, 1) = \left(2x + c_1, \frac{1}{2}x^2 + y + c_2\right)\Big|_{(0,1)} = (c_1, 1 + c_2) \implies c_1 = 0, \quad c_2 = -1,$$

and thus we need $f_1(x,y) = 2x$ and $f_2(x,y) = \frac{1}{2}x^2 + y - 1$. And now, by the inverse function theorem since, f does indeed provide a local coordinate system about $(0,1)$ since

$$\mathsf{d}f_{(0,1)} = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix}\Bigg|_{(0,1)} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

is invertible. □

Problem 5.

See problem 6 of 2005, Fall.

Problem 6.

Since $x^2 + y^2 + z^2 = 1$ on S^2, then by a simple calculation $\mathsf{d}\omega = 3\mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z$ on S^2, whereby

$$\int_{\mathsf{S}^2} \omega = \int_{\mathsf{B}^3} \mathsf{d}\omega = 3\int_{\mathsf{B}^3} \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z = 3\mathsf{vol}(\mathsf{B}^3) = 4\pi$$

by Stokes. □

Problem 7.

Remark. There's a mistake in the problem statement. We wish to show that the space of points $x \in \mathbb{R}^m$ such that $M \cap (\{x\} \times \mathbb{R}^n)$ is infinite has measure 0.

Let $\iota : M \hookrightarrow \mathbb{R}^m \times \mathbb{R}^n$ and $\pi : \mathbb{R}^m \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^m$ be the canonical inclusion and projection maps, respectively, and let $f := \pi \circ \iota : M \to \mathbb{R}^m$.

$$\begin{array}{ccc} M & \overset{\iota}{\hookrightarrow} & \mathbb{R}^m \times \mathbb{R}^n \\ & \underset{f}{\searrow} & \downarrow \pi \\ & & \mathbb{R}^m \end{array}$$

Let $x \in \mathbb{R}^m$ be a regular value of f. Then for any $y \in f^{-1}(x)$, the map $\mathsf{d}f_y : \mathsf{T}_y M \to \mathsf{T}_x \mathbb{R}^m$ is a surjective linear map of m-dimensional vector spaces, and thus a linear isomorphism. So by the inverse function theorem, there's an open neighborhood $U_y \subset M$ of y such that $f\big|_{U_y} : U_y \to f(U_y)$ is a diffeomorphism. Now, $f^{-1}(x)$ is a closed subset of the compact manifold M, since $\{x\} \subset \mathbb{R}^m$ is closed, and thus $f^{-1}(x)$ is itself compact. Then the open cover $\{U_y\}_{y \in f^{-1}(x)}$ of $f^{-1}(x)$ admits a finite subcover $\{U_{y_j}\}_{j=1}^k$. If $y \in f^{-1}(x)$ belongs to U_{y_j} for some $1 \leq j \leq k$, then we have $f\big|_{U_{y_j}}(y) = x = f\big|_{U_{y_j}}(y_j)$, and so $y = y_j$ since $f\big|_{U_{y_j}}$ is a diffeomorphism. Thus U_{y_j} contains no more than one element of $f^{-1}(x)$, for each $1 \leq j \leq k$, and since $\{U_{y_j}\}_{j=1}^k$ is a cover of $f^{-1}(x)$, it follows that $f^{-1}(x)$ is finite. Then

$$\begin{aligned} f^{-1}(x) &= \{(y_1,y_2) \in M \subset \mathbb{R}^m \times \mathbb{R}^n \mid f(y_1,y_2) = x\} = M \cap \{(y_1,y_2) \in \mathbb{R}^m \times \mathbb{R}^n \mid \pi(y_1,y_2) = x\} \\ &= M \cap \{(y_1,y_2) \in \mathbb{R}^m \times \mathbb{R}^n \mid y_1 = x\} = M \cap (\{x\} \times \mathbb{R}^n) \end{aligned}$$

is finite. So if $x \in \mathbb{R}^m$ is such that $M \cap (\{x\} \times \mathbb{R}^n)$ is infinite, then x is a critical value of f. By Sard, the critical values of f have measure 0. □