GEOMETRY TOPOLOGY QUALITFYING EXAM
SPRING 2013

Solve all of the problems that you can. Partial credit will be given for partial solutions.

(1) Consider the form

$$\omega = (x^2+x+y)dy\wedge dz$$

on $\mathbb{R}^3$. Let $S^2 = \{x^2+y^2+z^2=1\} \subset \mathbb{R}^3$ be the unit sphere, and $i\colon S^2 \to \mathbb{R}^3$ the inclusion.

stokes (a) Calculate $\int_{S^2} \omega$.

stokes (b) Construct a closed form α on $\mathbb{R}^3$ such that $i^*\alpha = i^*\omega$, or show that such a form α does not exist.

inverse fnl. thm. (2) Find all points in $\mathbb{R}^3$ in a neighborhood in which the functions $x, x^2+y^2+z^2-1, z$ can serve as a local coordinate system.

chart (3) Prove that the real projective space $\mathbb{R}P^n$ is a smooth manifold of dimension n.

$H^1_{dR}(S^n)=0$ (4) (a) Show that every closed 1-form on S^n, $n>1$ is exact.

$\mathbb{R}P^n \hookrightarrow S^n \twoheadrightarrow \mathbb{R}P^n$ (b) Use this to show that every closed 1-form on $\mathbb{R}P^n$, $n>1$ is exact.

homotope $5/S^1$ (5) Let X be the space obtained from $\mathbb{R}^3$ by removing the three coordinate axes. Calculate $\pi_1(X)$ and $H_*(X)$.

(6) Let $X = T^2 - \{p,q\}$, $p \neq q$ be the twice punctured 2-dimensional torus.

draw picture (a) Compute the homology groups $H_*(X,\mathbb{Z})$.

(b) Compute the fundamental group of X.

$\mathbb{Z}+\mathbb{Z} \hookrightarrow S^2$ (7) (a) Find all of the 2-sheeted covering spaces of $S^1 \times S^1$.

covering map (b) Show that if a path-connected, locally path connected space X has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic.

(8) (a) Show that if $f\colon S^n \to S^n$ has no fixed points then $\deg(f) = (-1)^{n+1}$.

(b) Show that if X has S^{2n} as universal covering space then $\pi_1(X) = \{1\}$ or $\mathbb{Z}_2$.

recall $\pi_1(X) \simeq G(S^{2n})$
and G acts freely on S^{2n}

Date: February 1, 2013.

Spring 2013

① $w = (x^2+x+y)\, dy \wedge dz$ on $\mathbb{R}^3$

Let $i : S^2 \hookrightarrow \mathbb{R}^3$

(a) calculate $\int_{S^2} \omega$

(b) Construct a closed form α on $\mathbb{R}^3 \ni i^*\alpha = i^*\omega$ or show that such a form α does not exist

Pf (a) $\displaystyle\int_{S^2} (x^2+x+y)\, dy \wedge dz \overset{\text{Stokes}}{=} \int_{B^2} d(x^2+x+y\, dy \wedge dz)$

$$= \int_{B^2} 2x\, dx \wedge dy \wedge dz + \int_{B^2} dx \wedge dy \wedge dz$$

Now B^2 is a symmetric domain & x an antisymmetric function

so $\displaystyle\int_{B^2} x\, dx \wedge dy \wedge dz = 0$ & $\displaystyle\int_{B^2} dx \wedge dy \wedge dz = Vol(B^2)$.

(b) if it did then $d\alpha = 0$, $i^* : H^k_{dR}(\mathbb{R}^3) \to H^k_{dR}(S^2)$

but if $\displaystyle\int_{S^2} i^*\alpha = \int_{S^2} i^*\omega = \int_{S^2} \omega \neq 0$

However $\displaystyle\int_{S^2} i^*\alpha = \int_{B} d(i^*\alpha) = \int_{B} i^*(d\alpha) = \int_{B} 0 = 0$

$\Rightarrow \Leftarrow$. So it cannot exist.

(2) Let $p(x,y,z) = (x, x^2+y^2+z^2-1, z)$

Where can it be a local coordinate system?

Pf consider $p_* = \begin{pmatrix} 1 & 0 & 0 \\ 2x & 2y & 2z \\ 0 & 0 & 1 \end{pmatrix}$

so $\det p_* = 2y = 0$ iff $y=0$.

so it workes aywhere on $\mathbb{R}^3 - \{y \text{ axis}\}$.

(4) (a) Show every closed 1-form on S^n; $n>1$ is exact

(b) use this to show that every closed 1 form on $\mathbb{R}P^n$ $n>1$ is exact.

Pf (a) Since $H^1_{dR}(S^n) = 0$ then if $d\eta = 0$ for $\eta \in \Omega^1(S^1)$

then $\eta \in H^1_{dR}(S^n)$. But then $[\eta] = 0 \Rightarrow \eta \in \operatorname{im} d: \Omega^0 \to \Omega^1$

so $\exists\ \beta \in \Omega^0 \ni d\beta = \eta$, so η is exact.

(b) consider $\mathbb{R}P^n \xhookrightarrow{p} S^n \xrightarrow{q} \mathbb{R}P^n$. Then $q \cdot p = id_{\mathbb{R}P^n}$

$[x] \longmapsto x \longmapsto [x]$

so $(q \cdot p)^* = id^*$. However $p^*: H^1_{dR}(S^n) \to H^1_{dR}(\mathbb{R}P^n)$

& $H^1(S^1) = 0 \Rightarrow p^* = 0$

But then $(q \circ p)^*: H^1_{dR}(\mathbb{R}P^n) \to H^1_{dR}(\mathbb{R}P^n)$ & $(q \circ p)^* = id$

so $H^1_{dR}(\mathbb{R}P^n) = 0$. Thus all closed 1 forms are exact.

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(3) Prove $\mathbb{R}P^n$ is smooth manifold of dim n.

Consider $\mathbb{R}^{n+1} - \{0\} \xrightarrow{\pi} \mathbb{R}P^n$. Then this map is continuous
$x \longmapsto [x]$ and open so it induces

a quotient topology on $\mathbb{R}P^n$.

Now, let $\tilde{U}_i \subseteq \mathbb{R}^{n+1} - \{0\} \ni x_i \neq 0$, & let $U_i = \pi(\tilde{U}_i)$

Then U_i is open and $\bigcup U_i \supseteq \mathbb{R}P^n$.

Now consider $\varphi_i : U_i \to \mathbb{R}^n \ni [x_1 \cdots x_{n+1}] \to \left(\frac{x_1}{x_i}, \ldots, \hat{x_i}, \ldots, \frac{x_n}{x_i}\right)$

then this map is conts & invertible w/

$\varphi^{-1}(x_1, , x_n) \to [x_1, \ldots, 1, \ldots, x_n]$

and smooth composition on intersections $U_i \cap U_j$.

So $\{U_i, \varphi_i\}$ are a chart to $\mathbb{R}^n$.

Lastly since $\mathbb{R}^{n+1}$ is Hausdorff & second countable via π

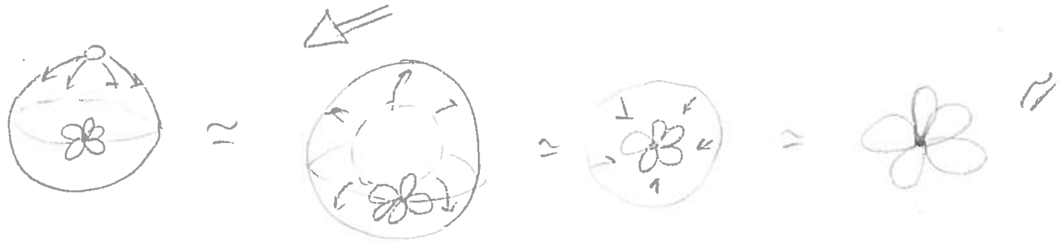
$\Rightarrow$ $\mathbb{R}P^n$ is also so $\mathbb{R}P^n$ is a n-dim smooth manifold.

⑤ Let $X = \mathbb{R}^3 -$ all axis. Find $\pi_1(X)$ & $H_*(X)$

Pf $\mathbb{R}^3 - 3$ axis $\Rightarrow$

$= S^1 - 6$ pts $= \bigvee^5 S^1$

wedge of 5 circles.



so $\pi_1(X) = \overset{5}{*}\,\mathbb{Z}$

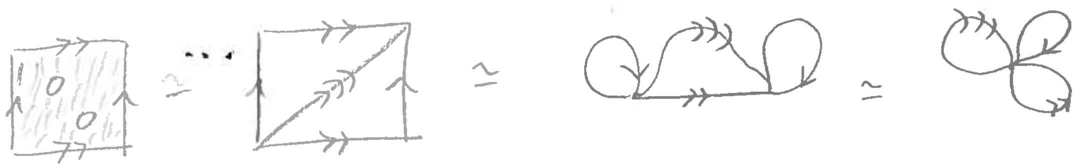
& $H_*(X) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}^5 & k=1 \\ 0 & \text{else.} \end{cases}$

⑥ Let $X = T^2 - \{p,q\}$ $p \neq q$.

(a) find $H_*(X, \mathbb{Z})$

(b) $\pi_1(X)$.

Pf $X =$



$\simeq \bigvee^3 S^1$

so $H_*(\bigvee^3 S^1) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}^3 & k=1 \\ 0 & \text{else} \end{cases}$ $\pi_1(X) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

(7) (a) Find all twosheeted covering spaces of $S^1 \times S^1$.
(b) Show if $\pi_1(X)$ is finite then $\forall f : X \to S^1$ is nullhomotopic.

Pf (a) want homomorphisms from $\mathbb{Z} \times \mathbb{Z} \to S_2$
$\Rightarrow \exists$ 4 such maps. $\Rightarrow$ 4 two sheeted connected covers.

(b) if $\pi_1(X)$ is finite then since $f_*(\pi_1(X)) \leq \pi_1(S^1) = \mathbb{Z}$
& the only finite subgroup of $\mathbb{Z}$ is $\{0\} \Rightarrow \pi_1(X) = 0$.

Thus $\exists$ lift
$$\begin{array}{ccc} & \overset{\tilde{f}}{\dashrightarrow} & \mathbb{R} \\ & & \downarrow p \\ X & \xrightarrow{f} & S^1 \end{array} \Rightarrow p \circ \tilde{f} = f.$$

But $\mathbb{R}$ is contractible so $\tilde{f} \simeq$ constant map (homotopic) $\Rightarrow p \circ \tilde{f}$ homotopic to constant map. $\Rightarrow f \simeq e$.

(8) (a) if $f : S^n \to S^n$ has no fixed points then $\deg(f) = (-1)^{n+1}$
(b) Show if X has S^{2n} as a universal cover then $\pi_1(X) = \{1\}$ or $\mathbb{Z}/2$.

Pf (a) if $f(x) \neq x \therefore f \simeq \alpha : x \mapsto -x \Rightarrow \deg f = \deg \alpha = (-1)^{n+1}$
(b) if S^{2n} is the universal cover then $\pi_1(X) \simeq G(S^{2n})$
the deck transformations of S^{2n}. However $G(S^{2n})$ acts freely on
$S^{2n} \Rightarrow G(S^{2n}) = \mathbb{Z}/2\mathbb{Z}$ or $\{1\}$. $\Rightarrow \pi_1(X) = \{1\}$ or $\mathbb{Z}/2\mathbb{Z}$
To see why consider $\gamma \in G(S^{2n})$ so $\gamma : S^{2n} \to S^{2n}$. Then $\deg \gamma = \pm 1$
so $\exists$ homomorphism $G(S^{2n}) \to \{\pm 1\} = \mathbb{Z}/2\mathbb{Z}$. Now each γ has no fixed
points so $\deg \gamma = (-1)^{2n+1} = -1 \Rightarrow G(S^{2n}) = \mathbb{Z}/2\mathbb{Z}$ (unless S^{2n} consists
of only one point).