Spring 2013 $\boxed{\#1}$

$$\omega = (x^2+x+y)\,dy\wedge dz \quad \text{in } \mathbb{R}^3$$

(a) Calculate $\int_{S^2} \omega$

(b) closed form α on $\mathbb{R}^3$ s.t. $i^*\alpha = i^*\omega$?

(a) $d\omega = (2x+1)\,dx\wedge dy\wedge dz$. Stokes gives

$$\int_{S^2} \omega = \int_{B^3} d\omega = \int_{B^3} 2x \, dx\,dy\,dz + \mathrm{vol}(B^3)$$

$$= \boxed{\mathrm{vol}(B^3)},$$ since B^3 is symetric across the x-axis

(b) Assume $\exists$ α s.t. $d\alpha = 0$ and $i^*\alpha = i^*\omega$.

Then $$\int_{S^2} i^*\alpha = \int_{S^2} i^*\omega = \int_{S^2} \omega \neq 0$$

But $$\int_{S^2} i^*\alpha = \int_{B^3} d(i^*\alpha) = \int_{B^3} i^*(d\alpha) = 0$$

$\Rightarrow\Leftarrow$.

Spring 2013 $\boxed{\#2}$

Find all points in $\mathbb{R}^3$ s.t. $\exists$ nbd of the point s.t. $(x, x^2+y^2+z^2-1, z)$ can serve as a local coordinate system.

Consider points of the form $(x,0,z) \in \mathbb{R}^3$.

Let U be a nbd of $(x_0, 0, z_0) =: P$. U contains a ball centered at $(x_0, 0, z_0)$, call it $B_\varepsilon(P)$, radius $\varepsilon > 0$.

Then $(x_0, -\frac{\varepsilon}{2}, z_0)$ and $(x_0, \frac{\varepsilon}{2}, z_0) \in B_\varepsilon(P)$.

But both are mapped to $(x_0, x_0^2 + \frac{\varepsilon^2}{4} + z_0^2 - 1, z_0)$

So this cannot serve as a local coordinate system.

For all other points $(x_0, y_0, z_0) \in \mathbb{R}^3$, $y_0 \neq 0$, we can find small ball s.t. y's are all the same sign inside the ball

$\phi: U \to \mathbb{R}^3$, $\phi(x,y,z) = (x, x^2+y^2+z^2-1, z)$

$\phi^{-1}: \phi(U) \to U$, $\phi^{-1}(u,v,w) = (u, \sqrt{v+1-u^2-w^2}, w)$

ϕ injective? if $x_0^2+y_0^2+z_0^2-1 = x_1^2+y_1^2+z_1^2-1$

and $x_0 = x_1$, $z_0 = z_1$ then $y_0^2 = y_1^2$ and if

$\operatorname{sgn}(y_0) = \operatorname{sgn}(y_1)$ then $y_0 = y_1$, so yes.

Thus ϕ is a homeomorphism onto $\phi(U)$

and we are done.

Spring 2013 $\boxed{\#3}$

Prove $\mathbb{RP}^n$ smooth mfd of dim'n n.

Let $q: S^n \longrightarrow \mathbb{RP}^n$ be the quotient map $q(x) = [x]$. Then let $U_i = q(V_i^{+/-})$ where $V_i^{+/-} = \{(x_1, \dots, x_{n+1}) \in S^n \subset \mathbb{R}^{n+1} : x_i \gtrless 0\}$. Note $q(V_i^-) = q(V_i^+)$

$\{U_i\}_1^n$ cover $\mathbb{RP}^n$. Let $\phi_i : U_i \longrightarrow \mathbb{R}^n$

$$\phi_i([x]) = \left(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_{n+1}}{x_i}\right)$$

well defined b/c $\left(\frac{-x_1}{-x_i}, \dots, \frac{-x_{i-1}}{-x_i}, \frac{-x_{i+1}}{-x_i}, \dots, \frac{-x_{n+1}}{-x_i}\right) =$

and continuous, and inverse

$$\phi_i^{-1}(u_1, \dots, u_n) = [u_1, \dots, u_{i-1}, 1, u_{i+1}, \dots, u_n]$$

which is also continuous. So homeo with $\mathbb{R}^n$

Hence $\mathbb{RP}^n$ is a smooth mfd of dim n.

Spring 2013 $\boxed{\#4}$

(a) Every closed 1-form on S^n, $n>1$ exact

(b) (a) $\Rightarrow$ every closed 1-form on $\mathbb{RP}^n$, $n>1$ exact.

(a) $H^1_{dR}(S^n) \approx H_1(S^n) = 0 \quad \forall\, n>1.$

$\Rightarrow$ closed 1-forms are exact.

(b) Let $f: \mathbb{RP}^n \longrightarrow S^n$ be a section of the quotient map $q: S^n \longrightarrow \mathbb{RP}^n$

Then $Id = q \circ f : \mathbb{RP}^n \xrightarrow{f} S^n \xrightarrow{q} \mathbb{RP}^n$ induces

$$f_* \circ q_* : H^1_{dR}(\mathbb{RP}^n) \xrightarrow{q_*} \underset{0}{H^1_{dR}(S^n)} \xrightarrow{f_*} H^1_{dR}(\mathbb{RP}^n)$$

the identity. Then f_* must be surjective and $H^1_{dR}(\mathbb{RP}^n) = 0$.

i.e. closed 1-forms exact.

Spring 2013 $\boxed{\#5}$

$X = \mathbb{R}^3 - \text{axes}$, Compute $\pi_1(X)$, $H_*(X)$.

$$X \simeq B^3 - \text{axes} \simeq S^2 - \text{axes} = S^2 - \{6 \text{ points}\}$$
$$\simeq D^2 - \{5 \text{ points}\}$$
$$\simeq (S^1)^{\vee 5}$$

$\Rightarrow \boxed{\pi_1(X) = F_5}$, free group on 5 elts.

$$\boxed{H_n(X) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}^{\oplus 5} & n=1 \\ 0 & \text{else} \end{cases}}$$

$r: X \longrightarrow S^2 - \text{axes}$ deformation retract.

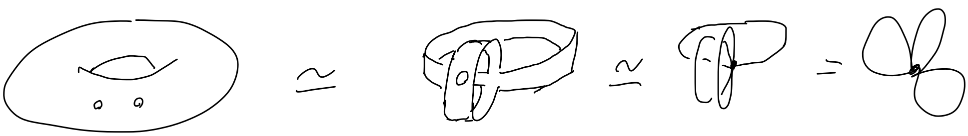
$$r(x) = \frac{x}{|x|}$$

Spring 2013 $\boxed{\#6}$

$$X = T^2 - \{p, q\}, \quad p \neq q$$

Compute $H_*(X; \mathbb{Z})$, $\pi_1(X)$.

$$T^2 - \{p,q\} \simeq S^1 \vee S^1 \vee S^1$$



So

$$\boxed{\begin{aligned} &\pi_1(X) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z} = F_3 \\ &H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{else} \end{cases} \end{aligned}}$$

Spring 2013 $\boxed{\#7}$

(a) Find all 2-sheeted covering spaces of $S^1 \times S^1$.

(b) X path connected, locally path connected, $\pi_1(X)$ finite, $f: X \to S^1 \Rightarrow f$ null homotopic.

(a) ~~Let $\tilde{T}$ be a 2-sheeted covering space of $S^1 \times S^1$.~~

~~$p: \tilde{T} \to T := S^1 \times S^1$.~~

~~$p_*: \pi_1(\tilde{T}) \to \pi_1(T) = \mathbb{Z} \times \mathbb{Z}$~~

~~$p_*(\pi_1(\tilde{T}))$ has index 2 in $\mathbb{Z} \times \mathbb{Z}$~~

$$\mathrm{Hom}\left(\pi_1(S^1 \times S^1), \Sigma_2\right) = \{\text{2-sheeted covering spaces}\}$$

$\Sigma_2 = \{(), (12)\} \approx \mathbb{Z}_2$.

Let $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}_2$ be a homomorphism.

f determined by $f((0,1))$ and $f((1,0))$.

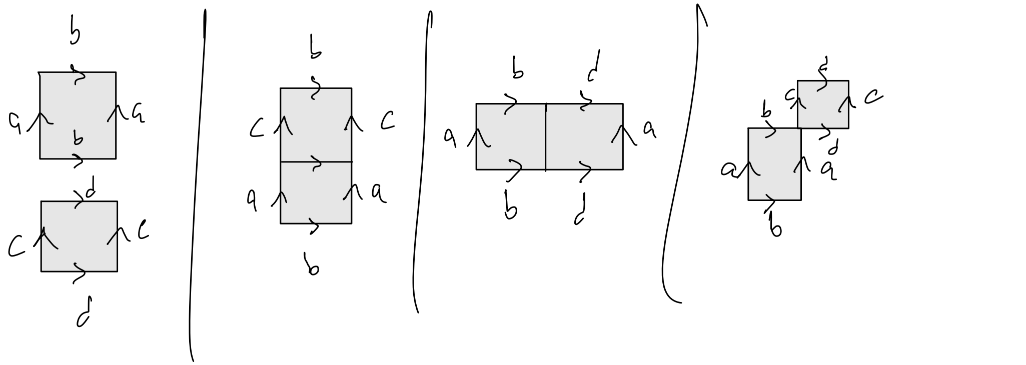
- $f((0,1)) = 0$ and $f((1,0)) = 0$

$= 1 \qquad = 0$

$= 0 \qquad = 1$

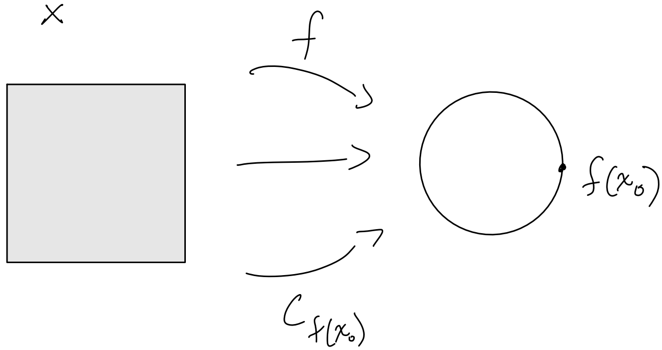
$= 1 \qquad = 1$

Each of these four are different 2-sheeted coverings, corresponding to



(b) X path connected, locally path connected, $\pi_1(X)$ finite, $f: X \to S^1 \Rightarrow f$ null homotopic.

$\overset{?}{\exists} H: I$



$H(x,t) =$

$$f_* : \pi_1(X) \longrightarrow \pi_1(S^1) \text{ hom}$$

$\Rightarrow$ $f_* = 0$ - map b/c any element in $\pi_1(X)$ has finite order and the only elt with finite order in $\pi_1(S^1) = \mathbb{Z}$ is 0.

$\Rightarrow$ f lifts to some $\tilde{f} : X \longrightarrow \mathbb{R}$ since $f_*(\pi_1(x)) = 0 \subset p_*(\pi_1(\mathbb{R})) = 0$ for universal covering map $p : \mathbb{R} \to S^1$.

Then $\tilde{f} \simeq c_0$ constant map by $H(x,t) = \tilde{f}(x)(1-t)$. $\Rightarrow p \circ \tilde{f} \simeq c_{p(0)}$

Then $f = p \circ \tilde{f} \simeq c_{p(0)}$.

I.e. f nullhomotopic.

Spring 2013 $\boxed{\#8}$

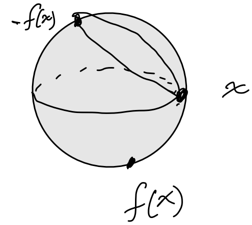
(a) $f: S^n \to S^n$ no fixed point $\Rightarrow$ $\deg(f) = (-1)^{n+1}$.

(b) X has S^{2n} as a universal covering space

$\Rightarrow$ $\pi_1(X) = \{1\}$ or $\mathbb{Z}_2$.

(a) $f: S^n \to S^n$ no fixed point.

Then $\exists$ straight line path γ_x from x to $-f(x)$ that doesn't pass through the origin. Then $\frac{\gamma_x}{|\gamma_x|}$ is a path from x to $-f(x)$ on S^n.



This defines a homotopy

$H: S^n \times I \to S^n$

$H(x,t) = \frac{\gamma_x(t)}{|\gamma_x(t)|}$ between Id

and $\alpha \circ f$, w/ α the antipodal map

Hence $\deg(\alpha)\deg(f) = \deg(\alpha \circ f) = \deg(Id) = 1$.

We know $\deg(\alpha) = (-1)^{n+1}$, so

$\deg(f)$ also $= (-1)^{n+1}$.

(b) X has S^{2n} as a universal covering space

$\Rightarrow$ $\pi_1(X)$ must be $\{1\}$ or $\mathbb{Z}_2$?

Since S^{2n} the universal cover of X,

$\pi_1(X) = G(S^{2n})$ the group of deck transformations on S^{2n}. This group acts freely on S^{2n}. That means

$\forall f \in G(S^{2n})$ nontrivial, $f: S^{2n} \to S^{2n}$ has no fixed points. $\overset{(a)}{\Rightarrow}$ $\deg(f) = (-1)^{2n+1} = -1$ $\forall f \in G(S^{2n})$ non trivial. Thus we define a map $d: G(S^{2n}) \longrightarrow \{\pm 1\}$ sending f to $\deg(f)$. d is a homomorphism because $\deg(fg) = \deg(f)\deg(g)$. And $\ker(d)$ = trivial elt in $G(S^{2n})$ by what we said above. Thus d injective homomorphism

$\Rightarrow$ $\pi_1(X) = G(S^{2n}) \subset \mathbb{Z}_2$.