2013, Spring

Problem 1.

(a) By Stokes,

$$\int_{\mathsf{S}^2} \omega = \int_{\mathsf{B}^3} \mathsf{d}\omega = \int_{\mathsf{B}^3} (2x+1)\mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z = 2\int_{\mathsf{B}^3} x\mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z + \int_{\mathsf{B}^3} \mathsf{d}x \wedge \mathsf{d}y \wedge \mathsf{d}z.$$

The first integral on the right vanishes since x is an odd function and B^3 is symmetric about 0, and so $\int_{\mathsf{S}^2} \omega = \mathsf{vol}(\mathsf{B}^3) = 4\pi/3$. □

(b) If $\alpha \in \Omega^2(\mathbb{R}^3)$ is a closed form with $i^*\alpha = i^*\omega$, then $\int_{\mathsf{S}^2} i^*\alpha = \int_{\mathsf{S}^2} i^*\omega = \int_{\mathsf{S}^2} \omega = 4\pi/3$, but also

$$\int_{\mathsf{S}^2} i^*\alpha = \int_{\mathsf{B}^3} \mathsf{d}(i^*\alpha) = \int_{\mathsf{B}^3} i^*(\mathsf{d}\alpha) = \int_{\mathsf{B}^3} i^*0 = 0,$$

a contradiction. □

Problem 2.

The given functions serve as local coordinates for any point $(x_0, y_0, z_0) \in \mathbb{R}^3$ about which the chart map $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\phi(x, y, z) := (x, x^2 + y^2 + z^2 - 1, z)$ is a local diffeomorphism. By the inverse function theorem, this condition is equivalent to the following differential being a linear isomorphism,

$$\mathsf{d}\phi_{(x_0,y_0,z_0)} = \begin{pmatrix} 1 & 0 & 0 \\ 2x_0 & 2y_0 & 2z_0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This matrix is invertible exactly when $y_0 \neq 0$, and so ϕ is a coordinate chart on $\mathbb{R}^3 \setminus \{y = 0\}$. □

Problem 3.

We define $\mathbb{R}\mathsf{P}^n$ as the quotient of $\mathbb{R}^{n+1} \setminus 0$ by the relation $x \sim cx$ for all $x \in \mathbb{R}^{n+1} \setminus 0$ and $c \in \mathbb{R} \setminus 0$. Equipping $\mathbb{R}\mathsf{P}^n$ with the quotient topology, it inherits the Hausdorff and second countable properties of $\mathbb{R}^{n+1} \setminus 0$. Now, consider the atlas $\{(U_j, \phi_j)\}_{j=0}^n$ for $\mathbb{R}\mathsf{P}^n$, where for each $1 \leq j \leq n$, we define $U_j := \{x_j \neq 0\} \subset \mathbb{R}\mathsf{P}^n$ and $\phi_j : U_j \to \mathbb{R}^n$,

$$\phi_j([x_0 : \cdots : x_n]) := \left(\frac{x_0}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_n}{x_j} \right).$$

This chart map is clearly continuous, and has continuous inverse given by

$$\phi_j^{-1}(y_0, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n) := [y_0 : \cdots : y_{j-1} : 1 : y_{j+1} : \cdots : y_n].$$

Moreover if $U_i \cap U_j \neq \varnothing$ for some $1 \leq i, j \leq n$, then the composite $\phi_i \circ \phi_j^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is easily seen to be smooth. Thus $\mathbb{R}\mathsf{P}^n$ is indeed an n-manifold. □

Problem 4.

(a) Let $n \in \mathbb{N}$. Seeing as $\mathsf{H}^1_{\mathsf{dR}}(\mathsf{S}^n) \cong 0$, if $\omega \in \Omega^1(\mathsf{S}^n)$ is closed, then $\omega \in \mathsf{ker}(\mathsf{d}^1) = \mathsf{im}(\mathsf{d}^0)$. □

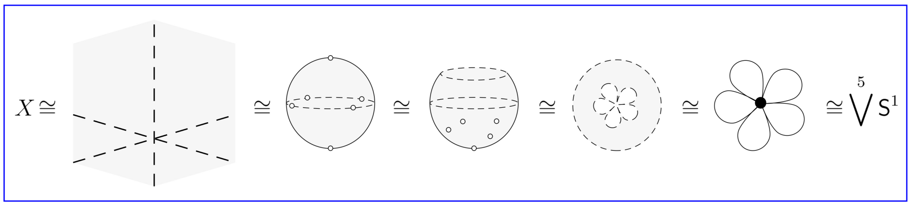
(b) Since $\mathbb{R}\mathsf{P}^n$ is the quotient of S^n by an action of $\mathbb{Z}_2$, we have a canonical projection $\pi : \mathsf{S}^n \twoheadrightarrow \mathbb{R}\mathsf{P}^n$ and a noncanonical inclusion $\iota : \mathbb{R}\mathsf{P}^n \hookrightarrow \mathsf{S}^n$. Let $\omega \in \Omega^1(\mathbb{R}\mathsf{P}^n)$ be closed. Then $\pi^*\omega \in \Omega^1(\mathsf{S}^n)$ is closed since $\mathsf{d}(\pi^*\omega) = \pi^*(\mathsf{d}\omega) = \pi^*0 = 0$. So by part (a), there's some $f \in \Omega^0(S^n)$ with $\mathsf{d}f = \pi^*\omega$. But then

$$\mathsf{d}(\iota^* f) = \iota^*(\mathsf{d}f) = \iota^*(\pi^*\omega) = (\pi \circ \iota)^*\omega = \omega,$$

where $\iota^* f \in \Omega^0(\mathbb{R}\mathsf{P}^n)$, and thus ω is exact. □

Problem 5.

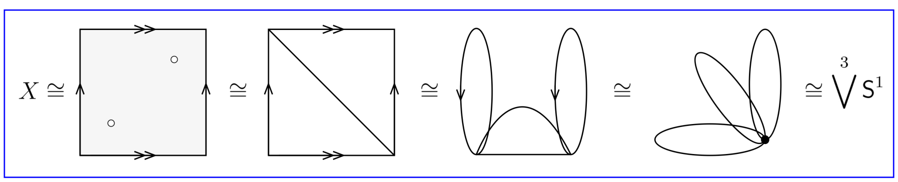
Along each axis, we squish $\mathbb{R}^3$ towards the points lying on the unit sphere, until we're left with a sphere with six points missing (corresponding to the intersections of $\mathbb{R}^3$ with the missing axes). We then stretch out one of these points while moving the remaining missing points close to one another. We're now left with an open disc with five missing points stuck together in the middle. Stretching out each of these missing points and pushing the outside of the disc toward their borders yields a wedge of five circles.



Hence $\pi_1(X) \cong \pi_1\left(\bigvee^5 \mathsf{S}^1\right) \cong \mathsf{F}_5$ and $\mathsf{H}_j(X) \cong \mathsf{H}_j\left(\bigvee^5 \mathsf{S}^1\right) \cong \begin{cases} \mathbb{Z} & j=0, \\ \mathbb{Z}^{\oplus 5} & j=1, \\ 0 & \text{else.} \end{cases}$ $\square$

Problem 6.

Viewing the torus as a square with edges identified, upon removing two points, we stretch each missing point out into a triangular region. This leaves the frame of the square together with a diagonal. Identifying the appropriate edges yields a wedge of three circles, as shown.



Then $\mathsf{H}_j(X) \cong \mathsf{H}_j\left(\bigvee^3 \mathsf{S}^1\right) \cong \begin{cases} \mathbb{Z} & j=0, \\ \mathbb{Z}^{\oplus 3} & j=1, \\ 0 & \text{else.} \end{cases}$ and $\pi_1(X) \cong \pi_1\left(\bigvee^3 \mathsf{S}^1\right) \cong \mathsf{F}_3$, by van Kampen. $\square$

Problem 7.

(a) The 2-sheeted covers of $\mathsf{S}^1 \times \mathsf{S}^1$ are classified by the index-2 subgroups of $\pi_1(\mathsf{S}^1 \times \mathsf{S}^1) \cong \mathbb{Z}^{\oplus 2}$. There are three such covers, corresponding to the subgroups $(2\mathbb{Z}) \oplus \mathbb{Z}, \mathbb{Z} \oplus (2\mathbb{Z})$, and $\mathsf{ker}(f)$ for $f : \mathbb{Z}^{\oplus 2} \to \mathbb{Z}_2$ the homomorphism given by $f(x, y) := [x + y]$. $\square$

(b) Denote by $\pi : \mathbb{R} \twoheadrightarrow \mathsf{S}^1$ the universal cover of S^1, and let $f : X \to \mathsf{S}^1$ be a continuous map. Then we have an induced homomorphism $f_* : \pi_1(X) \to \pi_1(\mathsf{S}^1)$, which must be trivial since $\pi_1(X)$ has torsion (it's finite) while $\pi_1(\mathsf{S}^1) \cong \mathbb{Z}$ doesn't. Then since $\mathbb{R}$ is simply connected, we have $f_*(\pi_1(X)) \cong 1 \subset \pi_*(\pi_1(\mathbb{R})) \cong 1$, and so there exists a lift

$$\begin{tikzcd} & \mathbb{R} \arrow[d, two heads, "\pi"] \\ X \arrow[ur, dashed, "\exists\, \tilde{f}"] \arrow[r, "f"] & \mathsf{S}^1. \end{tikzcd}$$

Again since $\mathbb{R}$ is simply connected, we may choose a homotopy $\{h_t : X \to \mathbb{R}\}_{0\le t\le 1}$ with $h_0 = \tilde{f}$ and $h_1 = c$ for some constant map $c : X \to \mathbb{R}$. Then $\{\pi \circ h_t\}_{0\le t\le 1}$ is a homotopy with $\pi \circ h_0 = \pi \circ \tilde{f} = f$, and $\pi \circ h_1 = \pi \circ c$ (a constant map). □

Problem 8.

(a) See problem 3 of 2014, Fall.

(b) Suppose S^{2n} is the universal cover of X, and denote by $G \cong \pi_1(X)$ its group of deck transformations. If $G \cong 1$ then we're done, so assume now that G is nontrivial. The action of G on S^{2n} is free since S^{2n} is path connected; so any $f \in G \setminus 1$ is a homeomorphism $\mathsf{S}^{2n} \to \mathsf{S}^{2n}$ with no fixed points, and $\mathsf{deg}(f) = (-1)^{2n+1} = -1$ by (a). Choosing some $g, h \in G \setminus 1$, we have that

$$\mathsf{deg}(g^2) = [\mathsf{deg}(g)]^2 = (-1)^2 = 1, \quad \mathsf{deg}(gh) = \mathsf{deg}(g)\mathsf{deg}(h) = (-1)(-1) = 1,$$

so $g^2 = 1 = gh$ by the above. This gives $g = h$, and we conclude that G consists of the identity element and a single nontrivial element $g \in G$ with $g^2 = 1$. As such, $G \cong \mathbb{Z}_2$. □