Geometry/Topology Qualifying Exam

Spring 2012

Solve all **SEVEN** *problems. Partial credit will be given to partial solutions.*

Inv. fn. thm
compactness

1. (10 pts) Prove that a compact smooth manifold of dimension n cannot be immersed in $\mathbb{R}^n$.

2. (10 pts) Let $\Sigma_{1,1}$ be the compact oriented surface with boundary, obtained from $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates (x, y) by removing a small disk $\{(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 = \frac{1}{100}\}$.

   $\Sigma_{1,1} = S^1 \vee S^1$

   (a) Compute the homology of $\Sigma_{1,1}$.

   Mayer Vietoris

   (b) Let Σ_2 denote a closed oriented surface of genus 2. Use your answer from (a) to compute the homology of Σ_2.

   recall
   $e_1 \times e_2 = \vec{n}$

3. (10 pts) Let S be an oriented embedded surface in $\mathbb{R}^3$ and ω be an area form on S which satisfies $\omega(p)(e_1, e_2) = 1$ for all $p \in S$ and any orthonormal basis (e_1, e_2) of T_pS with respect to the standard Euclidean metric on $\mathbb{R}^3$. If (n_1, n_2, n_3) is the unit normal vector field of S, then prove that
$$\omega = n_1 dy \wedge dz - n_2 dx \wedge dz + n_3 dx \wedge dy,$$
where (x, y, z) are the standard Euclidean coordinates on $\mathbb{R}^3$.

4. (10 pts) Consider the space $X = M_1 \cup M_2$, where M_1 and M_2 are Möbius bands and $M_1 \cap M_2 = \partial M_1 = \partial M_2$. Here a *Möbius band* is the quotient space $([-1,1]\times[-1,1])/((1,y) \sim (-1,-y))$.

   van kampen
   picture.

   (a) Determine the fundamental group of X.

   (b) Is X homotopy equivalent to a compact orientable surface of genus g for some g?

5. (10 pts) Determine all the connected covering spaces of $\mathbb{RP}^{14} \vee \mathbb{RP}^{15}$.

6. (10 pts) Let $f : M \to N$ be a smooth map between smooth manifolds, X and Y be smooth vector fields on M and N, respectively, and suppose that $f_*X = Y$ (i.e., $f_*(X(x)) = Y(f(x))$ for all $x \in M$). Then prove that $f^*(\mathcal{L}_Y \omega) = \mathcal{L}_X(f^*\omega)$, where ω is a 1-form on N. Here $\mathcal{L}$ denotes the Lie derivative.

   cartan

7. (10 pts) Consider the linearly independent vector fields on $\mathbb{R}^4 - \{0\}$ given by:
$$X(x_1,x_2,x_3,x_4) = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} + x_4\frac{\partial}{\partial x_4}$$
$$Y(x_1,x_2,x_3,x_4) = -x_2\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial x_3} + x_3\frac{\partial}{\partial x_4}.$$

   frobenius
   thm.

   Is the rank 2 distribution orthogonal to these two vector fields integrable? Here orthogonality is measured with respect to the standard Euclidean metric on $\mathbb{R}^4$.

Spring 2012

1) Prove a compact smooth manifold of dim n cannot be immersed in $\mathbb{R}^n$.

Pf M - compact smooth w/ dim n.

So if $\varphi: M \to \mathbb{R}^n$ an immersion $\Rightarrow \varphi_* . T_pM \to T_q\mathbb{R}^n$ is injective.

$\Rightarrow \dim M = n \Rightarrow \dim T_pM = n \Rightarrow \varphi_*$ is an isomorphism.

Hence φ is a local diffeomorphism since $\varphi_{*p} \neq 0$ $\forall p \in T_pM$ so we can apply the inverse function theorem on $\varphi(M)$.

Since M is compact and φ is continuous $\varphi(M)$ is compact in $\mathbb{R}^n$. Thus to each $p \in \varphi(M)$ $\exists$ $U_i \ni p \in U_i \simeq \varphi^{-1}(U_i) = W_i$.

Now each W_i is open by continuity of φ. Now $\bigcup W_i$ is an open cover of M so by compactness $\exists$ finite subcover, so that $\varphi(M) = \varphi(\bigcup^k W_i) = \bigcup^k \varphi(W_i) = \bigcup^k U_i$

However each U_i is open $\Rightarrow \bigcup U_i$ is open in $\mathbb{R}^n$.

But M is compact $\Rightarrow \varphi(M)$ is compact. So $\varphi(M)$ is both closed & open $\Rightarrow \varphi(M) = \phi$ or $\varphi(M) = \mathbb{R}^n$.

if $\varphi(M) = \phi \Rightarrow M = \phi \Rightarrow \varphi_* : \phi \to T_q\mathbb{R}^n \Rightarrow T_q\mathbb{R}^n \simeq \phi \Rightarrow \in$.

if $\varphi(M) = \mathbb{R}^n \Rightarrow \mathbb{R}^n$ is compact $\Rightarrow \in$.

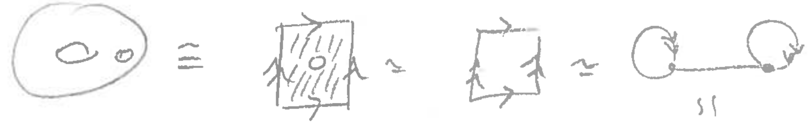
Thus such a map is impossible.

② $\Sigma_{1,1}$ is T^2 - small disk $= \{(x-\frac{1}{2})^2+(y-\frac{1}{2})^2=\frac{1}{100}\}$, $T^2 = \mathbb{R}^2/\mathbb{Z}^2$

(a) compute the homology of $\Sigma_{1,1}$

(b) let Σ_2 denote the closed oriented surface of genus 2. Compute $H_*(\Sigma_2)$

Pf since $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ & $\{(x-\frac{1}{2})^2+(y-\frac{1}{2})^2=\frac{1}{100}\} \cap \mathbb{Z}^2 = \emptyset$

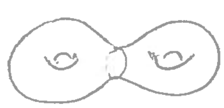
then $\Sigma_{1,1} \simeq T^2 - pt. \Rightarrow$ 

$\cong S^1 \vee S^1$

so $\Sigma_{1,1} \simeq S^1 \vee S^1$

(a) $\tilde{H}_k(S^1 \vee S^1) \simeq \tilde{H}_k(S^1) \oplus \tilde{H}_k(S^1)$ (since all are good pairs)

$$\Rightarrow \quad H_k(\Sigma_{1,1}) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}\oplus\mathbb{Z} & k=1 \\ 0 & \text{else} \end{cases}$$

(b) $\Sigma_2 =$  so let $A = T^2 - pt$ $\quad B = T^2 - pt$

$A \cup B = \Sigma_2 \qquad A \cap B \simeq S^1$

$\Rightarrow \cdots \to H_k(A\cap B) \to H_k(A) \oplus H_k(B) \to H_k(A\cup B) \to H_{k-1}(A \cap B) \to \cdots$

so $k \geq 3 \Rightarrow H_k(\Sigma_2) = 0$

$k \leq 2 \Rightarrow \quad 0 \xrightarrow{j_2} H_2(\Sigma_2) \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{i_1} \mathbb{Z}^2 \oplus \mathbb{Z}^2 \xrightarrow{j_1} H_1(\Sigma_2) \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{i_0} \mathbb{Z}^2$

$\downarrow j_0$

$H_0(\Sigma_2)$

$\downarrow \partial_0$

0

$H_0(\Sigma_2) = \mathbb{Z}$ by path conn.

$\ker j_2 = 0$, $\operatorname{im} j_2 = 0 = \ker \partial_2$

$\Rightarrow H_2(\Sigma_2) = \operatorname{im} \partial_2$

$\ker \partial_0 = H_0(\Sigma_2) = \mathbb{Z} \Rightarrow \operatorname{Im} j_0 = \mathbb{Z}$, $\ker j_0 = 0 \Rightarrow \operatorname{Im} i_0 = \mathbb{Z}$

$\Rightarrow \ker i_0 = 0 \Rightarrow \operatorname{Im} \partial_1 = 0$ so $H_1(\Sigma_2) = \ker \partial_1 = \operatorname{Im} j_1$

$\operatorname{Im} i_1 = 0 \Rightarrow \ker j_1 = 0 \Rightarrow \operatorname{Im} j_1 = \mathbb{Z}^4 \Rightarrow H_1(\Sigma_2) = \mathbb{Z}^4$

$H_1(A \cap B) \to H_1(A) \oplus H_1(B)$ $\qquad H_2(\Sigma_2) = \operatorname{im} \partial_2 = \ker i_1 = \mathbb{Z}$

$\alpha \longmapsto 0, 0$

$\ker i_1 = \alpha = \mathbb{Z}$

$\operatorname{Im} i_1 = 0$

$$\Rightarrow H_k(\Sigma_2) = \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}^4 & k = 1 \\ \mathbb{Z} & k = 2 \\ 0 & \text{else} \end{cases}$$

(3) S - oriented embedded surface in $\mathbb{R}^3$, ω - area form on S

$\Rightarrow \omega(p)(e_1, e_2) = 1 \quad \forall p$ w/ e_1, e_2 of T_pS

if (n_1, n_2, n_3) is the unit normal v. field of S prove.

$\omega = n_1 dy \wedge dz - n_2 dx \wedge dz + n_3 dx \wedge dy$.

~~Pf~~ let $w = w_1 dy \wedge dz - w_2 dx \wedge dz + w_3 dx \wedge dz$. Since S is a surface in $\mathbb{R}^3$, it has dim 2, so T_pS has dim 2

hence w must be a two form representable as above.

Now, if e_1, e_2 are an O.N.B for $T_pS \Rightarrow$ $e_1 \times e_2 = \vec{n} = (n_1, n_2, n_3)$

so if $e_1 = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}$ & $e_2 = b_1 \frac{\partial}{\partial x} + b_2 \frac{\partial}{\partial y} + b_3 \frac{\partial}{\partial z}$

$\Rightarrow (a_2 b_3 - a_3 b_2) = n_1$, $-(a_1 b_3 - a_3 b_1) = n_2$, $(a_1 b_2 - a_2 b_1) = n_3$

so then if $w_p(e_1, e_2) = 1$

$$w_p(e_1,e_2) = w_p\left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z},\ b_1\frac{\partial}{\partial x} + b_2\frac{\partial}{\partial y} ; b_3\frac{\partial}{\partial z}\right)$$

$$= w_1(p)\,dy\wedge dz\,(e_1,e_2) + w_2(p)\,dx\wedge dz\,(e_1,e_2) + w_3(p)\,dx\wedge dz$$

$$= w_1(p)\left[dy(e_1)\,dz(e_2) - dy(e_2)\,dz(e_1)\right] + w_2(p)\left[dx(e_1)\,dz(e_2) - dx(e_2)\,dz(e_1)\right]$$
$$+ w_3(p)\left[dx(e_1)\,dz(e_1) - dx(e_2)\,dz(e_1)\right]$$

$$= w_1(p)\left[a_2b_3 - b_2a_3\right] + w_2(p)\left[a_1b_3 - a_3b_1\right] + w_3(p)\left[a_1b_2 - a_2b_1\right]$$

$$= n_1w_1(p) - n_2w_2(p) + n_3w_3(p) = 1.$$

However $n_1^2 + n_2^2 + n_3^2 = 1 \Rightarrow w_1(p) = n_1 \quad w_2(p) = -n_2 \quad w_3(p) = n_3$

□

(4) $X = M_1 \cup M_2$ w/ $M \cap M_2 = \partial M_1 = \partial M_2$

(a) find $\pi_1(X)$

(b) is X homotopy equivalent to a compact orientable surface of genus g?

Pf a) $A = M_1 \quad B = M_2 \quad A\cap B \cong S^1 \quad A \cup B = X$, By van Kampen:

$$\begin{tikzcd} & \pi_1(A) \arrow[dr] & \\ \pi_1(A\cap B) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & \pi_1(A) * \pi_1(B) / N \\ & \pi_1(B) \arrow[ur] & \end{tikzcd}$$

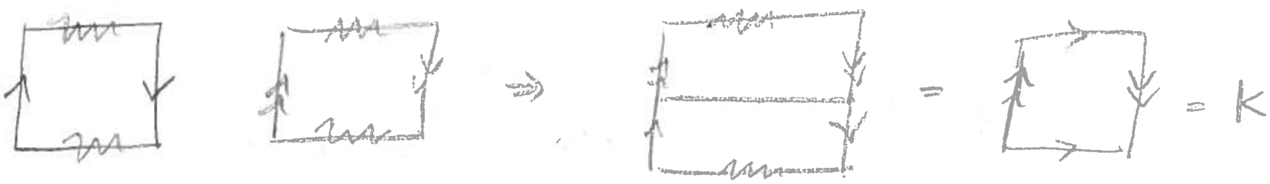
where $A \simeq S^1$
$B \simeq S^1$
so let $\pi_1(A) = \langle a \rangle$, $\pi_1(B) = \langle b \rangle$

$i_1: \langle \gamma \rangle \longrightarrow \langle a \rangle$
$\gamma \longmapsto a^2$

so $\pi_1(X) = \dfrac{\langle a,b \rangle}{\langle a^2b^{-2} \rangle} = \langle a,b \mid a^2 = b^2 \rangle$

$i_2: \langle \gamma \rangle \longmapsto \langle b \rangle$
$\gamma \longmapsto b^2$

(b)

No; 

so X is $\simeq$ klein bottle, which is non orientable.

(5) Determine all covering spaces of $\mathbb{RP}^{14} \vee \mathbb{RP}^{15}$

~~If~~ since $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2\mathbb{Z}$ for $n \geq 2$ then its universal cover is S^n.

Now, for any $X \vee Y$ the universal cover is an alternating ∞ wedge of both so for $\mathbb{RP}^{14} \vee \mathbb{RP}^{15}$ the universal cover is an infinite wedge of S^{14} & S^{15} $\Rightarrow$ 

Intermediate covers are in bijection w/ subgroups of $\pi_1(X) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ i.e. subgroups of... $\langle a, b \mid a^2 = b^2 = e \rangle$, so all words of the form $a^i b a b \dots a b^j$ w/ $i,j = 1,0,1,0.$ $\overset{bij}{\Longleftrightarrow}$ $\mathbb{N}$.

Clearly these correspond to alternating wedges of S^{14} & S^{15} of finite length.

7) Given X & Y linearly indep on $\mathbb{R}^4 - \{0\}$ is the rank 2 distribution orthogonal to these two vector fields integrable

Pf Using Frobenius' theorem we compute $[X, Y]$.

$$[X,Y] = \left[x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4},\ -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4}\right]$$

$$= \cancel{x_1 \frac{\partial}{\partial x_2}} - \cancel{x_2 \frac{\partial}{\partial x_1}} + \cancel{x_3 \frac{\partial}{\partial x_4}} - \cancel{x_4 \frac{\partial}{\partial x_3}}$$

$$- \left(\cancel{- x_2 \frac{\partial}{\partial x_1}} + \cancel{x_1 \frac{\partial}{\partial x_2}} - \cancel{x_4 \frac{\partial}{\partial x_3}} + \cancel{x_3 \frac{\partial}{\partial x_4}} \right) = 0$$

Thus $[X,Y] = 0 \Rightarrow \langle [X,Y], X \rangle = \langle [X,Y], Y \rangle = 0$

so by Frobenius $[X,Y]$ is orthogonal to X, Y $\Rightarrow$ it is a linear combination of span $\langle X, Y \rangle^{\perp}$, & hence integrable.

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