Spring 2012 $\boxed{\#1}$

M compact n-dim'l mfd. Claim: M cannot be immersed in $\mathbb{R}^n$.

Assume $f: M \to \mathbb{R}^n$ is an immersion

then $df_p: T_pM \to T_{f(p)}\mathbb{R}^n$ is injective $\forall\, p \in M$
$\mathbb{R}^n \to \mathbb{R}^n$

which mens it is surjective $\forall\, p \in M$ by dimension.
Thus f is also a submersion and therefore a local diffeomorphism.
Since f is smooth, $f(M)$ compact in $\mathbb{R}^n$.
Let $\{U_i\}_{i=1}^{k}$ be an open cover of M s.t.

$F|_{U_i}: U_i \to F(U_i)$ is a diffeomorphism $\forall\, i$.

Then it is also a homeomorphism $\forall\, i$, and therefore $F(U_i)$ open and $\bigcup_{i=1}^{k} F(U_i) = M$ is open in $\mathbb{R}^n$.

Then $\mathbb{R}^n = \mathbb{R}^n - M \sqcup M$ is the disjoint union of two open sets $\Rightarrow$ $\mathbb{R}^n$ disconnected $\Rightarrow\Leftarrow$.

Spring 2012 $\boxed{\#2}$

$\Sigma_{1,1} = T^2 -$ small disc.

(a) compute $H_n(\Sigma_{1,1})$

(b) $\Sigma_2 =$ closed, oriented surface of genus 2. Use (a) to compute $H_n(\Sigma_2)$

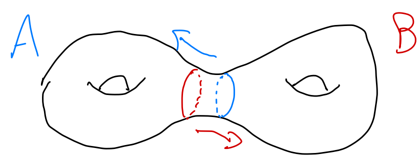
(a) $T^2 -$ small disc def. retracts to $S^1 \vee S^1$

So, $\tilde{H}_n(\Sigma_{1,1}) \approx \tilde{H}_n(S^1 \vee S^1) \approx \tilde{H}_n(S^1) \oplus \tilde{H}_n(S^1)$

$$= \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow H_n(\Sigma_{1,1}) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z} \oplus \mathbb{Z} & n=1 \\ 0 & \text{else} \end{cases}$$

(b) Let $A, B = 2$ overlapping punctured tori as in the diagram:



Then $A \cup B = \Sigma_2$,

$A, B \simeq \Sigma_{1,1}$ and $A \cap B \simeq S^1$.

Hence, M.V. gives $\cdots \to H_n(S^1) \to H_n(\Sigma_{1,1}) \oplus H_n(\Sigma_{1,1}) \to H_n(\Sigma_2) \to \cdots$

For $n \geq 3$ we have

$0 \to H_n(\Sigma_2) \to 0$ implying $H_n(\Sigma_2) = 0$.

Otherwise, we have

Since Σ_2 path connected

$$0 \to H_2(\Sigma_2) \to \mathbb{Z} \to \mathbb{Z}^4 \to H_1(\Sigma_2) \to \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z} \to \mathbb{Z} \to 0$$

Consider the map $H_1(A\cap B) \xrightarrow{i_*, j_*} H_1(A)\oplus H_1(B)$.

A 1-chain in $A\cap B$ is the boundary of a 2-chain in A or B $\Rightarrow$ i_*, j_* are 0-maps.

Hence, $\mathbb{Z}^4 \hookrightarrow H_1(\Sigma_2)$. Consider the map $H_0(A\cap B) \to H_0(A)\oplus H_0(B)$ is injective $\Rightarrow$ $H_1(\Sigma_2) \to \mathbb{Z}$ is the 0-map $\Rightarrow$ $\mathbb{Z}^4 \twoheadrightarrow H_1(\Sigma_2)$. Hence $H_1(\Sigma_2) = \mathbb{Z}^4$

$\mathbb{Z} \xrightarrow{\times 0} \mathbb{Z}^4$ also $\Rightarrow$ $H_2(\Sigma_2) \twoheadrightarrow \mathbb{Z}$ and the $0 \to H_2(\Sigma_2)$ $\Rightarrow$ $H_2(\Sigma_2) \hookrightarrow \mathbb{Z}$. Hence $H_2(\Sigma_2) = \mathbb{Z}$

In sum,

$$\boxed{H_n(\Sigma_2) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^4 & n = 1 \\ 0 & \text{else} \end{cases}}$$

Spring 2012 $\boxed{\#3}$

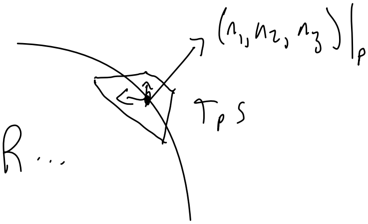
S oriented embedded surface in $\mathbb{R}^3$ and ω area form on S

$\omega_p(e_1, e_2) = 1 \quad \forall p \in S$ and any orthonormal basis (e_1, e_2) of T_pS

If (n_1, n_2, n_3) is the unit normal vect. field of S,

then prove that $\omega = n_1\, dy \wedge dz - n_2\, dx \wedge dz + n_3\, dx \wedge dy$.

$$\omega = P\, dy \wedge dz + Q\, dx \wedge dz + R\, dx \wedge dy \in \Omega^2(S)$$



$$\omega_p(e_1, e_2) = P(p)\,(dy \wedge dz)(e_1, e_2) + Q(p)\,(dx \wedge dz)(e_1, e_2) + R \ldots$$

$$= P(p)\left(e_{1y} e_{2z} - e_{1z} e_{2y}\right) + Q(p)\left(e_{1x} e_{2z} - e_{1z} e_{2x}\right)$$
$$+ R(p)\left(e_{1x} e_{2y} - e_{1y} e_{2x}\right) = 1$$

$$= P n_1 + Q n_2 + R n_3 = 1$$

$$e_1 \times e_2 = \det\begin{pmatrix} e_{1x} & e_{1y} & e_{1z} \\ e_{2x} & e_{2y} & e_{2z} \\ i & j & k \end{pmatrix}$$

$$\big(\overbrace{e_{1y} e_{2z} - e_{1z} e_{2y}}^{n_1}, \ldots$$

$$P n_1 + Q n_2 + R n_3 = n_1^2 + n_2^2 + n_3^2$$

$$n_1(P - n_1) + n_2(Q - n_2) + n_3(R - n_3) = 0$$

$\forall\, p \in S.$

$$dx \wedge \omega = dx \wedge P\, dy \wedge dz = \frac{\partial P}{\partial x}\, dx \wedge dy \wedge dz = 0$$

$$d\omega = \frac{\partial P}{\partial x} dx \wedge dy \wedge dz + \frac{\partial Q}{\partial y} dy \wedge dx \wedge dz + \frac{\partial R}{\partial z} dz \wedge dx \wedge dy$$

Spring 2012 $\boxed{\#4}$

$X = M_1 \cup M_2$ M_1 and M_2 mobius bands and

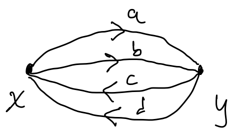
$M_1 \cap M_2 = \partial M_1 = \partial M_2$. Mobius band $= ([-1,1]\times[-1,1])/\sim$

$(1,y) \sim (-1,-y)$.

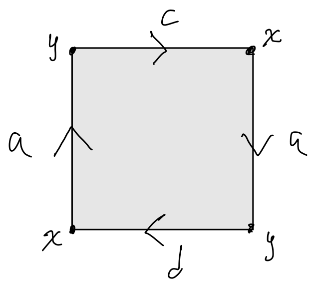
(a) compute $\pi_1(X)$ (b) no b/c $\simeq$ Klein bottle

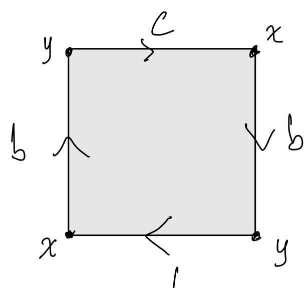
(b) $X \simeq$ compact orientable surface of genus g for some g?

As a CW complex, X has two 0-cells, x, y. It has four 1-cells a,b,c,d. So before attaching 2-cells it looks like



which is homotopy equivalent to $S^1 \vee S^1 \vee S^1$ and has fundamental gp $\mathbb{Z} * \mathbb{Z} * \mathbb{Z} = \langle ab^{-1}, ac, ad \rangle$





We attach two 2-cells along the paths $acad$ and $bcbd = ba^{-1}acba^{-1}ad$

$$= (ab^{-1})^{-1}ac(ab^{-1})^{-1}ad$$

Then $\pi_1(X) = \langle ab^{-1}, ac, ad \mid acad = (ab^{-1})^{-1}ac(ab^{-1})^{-1}ad = 1 \rangle$

$$= \langle u, v, w \mid vw = u^{-1}vu^{-1}w = 1 \rangle$$

$$= \langle u, v \mid u^{-1}vu^{-1}v^{-1} = 1 \rangle = \boxed{\langle u, v \mid v = uvu \rangle}$$

Spring 2012 $\boxed{\#5}$

Determine connected covering spaces of $\mathbb{RP}^{14} \vee \mathbb{RP}^{15}$

$\pi_1(\mathbb{RP}^{14} \vee \mathbb{RP}^{15}) = \pi_1(\mathbb{RP}^{14}) * \pi_1(\mathbb{RP}^{15})$.

For $n \geq 2$, S^n is a 2-sheeted universal cover of $\mathbb{RP}^n$ (since S^n simply connected). Thus, it's induced (trivial) image in $\pi_1(\mathbb{RP}^n)$ has index 2 $\Rightarrow \pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$

So $\pi_1(\mathbb{RP}^{14} \vee \mathbb{RP}^{15}) = \mathbb{Z}_2 * \mathbb{Z}_2 = \langle a, b \mid a^2 = b^2 = 1 \rangle$

The connected covering spaces of $\mathbb{RP}^{14} \vee \mathbb{RP}^{15}$ thus correspond to the subgroups of $\mathbb{Z}_2 * \mathbb{Z}_2$.

$a, b, ab, ba, aba, abab,$

$$\left\{ \begin{array}{l} \quad b,\ ba,\ bab,\ baba,\ \dots \\ 1 \\ \quad a,\ ab,\ aba,\ abab,\ \dots \end{array} \right\}$$

$\vdots$

$ababa$

aba

a

$\mathbb{Z}$ $\quad \dots\ baba \quad ba \quad | \quad ab \quad abab \dots$

b

bab

$babab$

$\vdots$

$\langle ab \cdots ba \rangle = \{1, ab \cdots ba\}$

$\mathbb{Z}_2$

$\langle ba \cdots ab \rangle =$

$$\langle ab \rangle = \left\{ \begin{matrix} 1, & ab, & abab, & ababab, & \dots \\ & ba, & baba, & baba, & \dots \end{matrix} \right\}$$

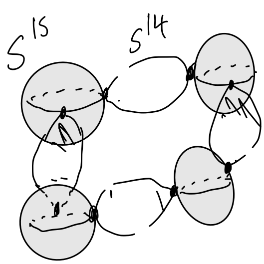
which has index 2

indeed a or b multiplied by $\langle ab \rangle$ gives you the rest of the group.

$$\langle \underbrace{ab \cdots ab}_{n(ab)\text{'s}} \rangle = \left\{ \begin{matrix} 1, & n(ab), & 2n(ab), & 3n(ab), & \dots \\ & n(ba), & 2n(ba), & 3n(ba), & \dots \end{matrix} \right\}$$

which has index $2n$

The "rings" of alternating S^{14}, S^{15} are the connected covers:



even-sheeted covers.

Spring 2012 $\boxed{\#6}$

$f: M \to N$ smooth, X, Y vect. fields on M, N.

$f_* X = Y$, i.e. $f_*(X(x)) = Y(f(x))$ $\forall x \in M$.

Claim: $f^* \mathcal{L}_Y \omega = \mathcal{L}_X (f^* \omega)$, $\omega \in \Omega^1(N)$.

$$\mathcal{L}_Y \omega = d(Y \lrcorner \omega) + Y \lrcorner d\omega$$

$$\begin{aligned}
(f^* \mathcal{L}_Y \omega)_x(v) &= (f^* d(Y \lrcorner \omega))_x(v) + (f^*(Y \lrcorner d\omega))_x(v) \\
&= d(Y \lrcorner \omega)_{f(x)}(df_x(v)) + (Y \lrcorner d\omega)_{f(x)}(df_x(v)) \\
&= \frac{d}{dt}\Big|_{t=0} \big((Y \lrcorner \omega)((f \circ \gamma)(t))\big) + d\omega_{f(x)}(Y(f(x)), df_x(v)) \\
&= \frac{d}{dt}\Big|_{t=0} \big(\omega(Y((f \circ \gamma)(t)))\big) + d\omega_{f(x)}(f_*(X(x)), df_x(v))
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_X (f^* \omega)_x(v) &= (d(X \lrcorner f^* \omega))_x(v) + (X \lrcorner d f^* \omega)_x(v) \\
&= \frac{d}{dt}\Big|_{t=0} \big((X \lrcorner f^* \omega)(\gamma(t))\big) + df^*\omega_x(X(x), v) \\
&= \frac{d}{dt}\Big|_{t=0} \big(f^*\omega(X(\gamma(t)))\big) + f^* d\omega_x(X(x), v) \\
&= \frac{d}{dt}\Big|_{t=0} \big(\omega(f_* X(\gamma(t)))\big) + d\omega_{f(x)}(f_*(X(x)), df_x(v)) \\
&= \frac{d}{dt}\Big|_{t=0} \big(\omega(X(f \circ \gamma(t)))\big) + d\omega_{f(x)}(f_*(X(x)), df_x(v))
\end{aligned}$$

where $\gamma: [-1,1] \to M$, $\gamma(0) = x$, $\gamma'(0) = v$

$\Rightarrow$ $f \circ \gamma: [-1,1] \to N$, $(f \circ \gamma)(0) = f(x)$, $(f \circ \gamma)'(0) = df_x v$

Spring 2012 $\boxed{\#7}$

X, Y vect. fields on $\mathbb{R}^4 - \{0\}$

$$X = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4}$$

$$Y = -x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2} - x_4 \frac{\partial}{\partial x_3} + x_3 \frac{\partial}{\partial x_4}.$$

Is the rank 2 distribution orthogonal to these two vect. fields integrable?

See Chapter 19 Lee Smooth mfds.

$$[X, Y]f = XYf - YXf$$

$$XYf = X\left(-x_2 \frac{\partial f}{\partial x_1} + x_1 \frac{\partial f}{\partial x_2} - x_4 \frac{\partial f}{\partial x_3} + x_3 \frac{\partial f}{\partial x_4}\right)$$

$$= -x_1 x_2 f_{11} + x_1 f_2 + x_1^2 f_{12} - x_1 x_4 f_{13} + x_1 x_3 f_{14}$$

$$- x_2 f_1 - x_2^2 f_{12} + x_1 x_2 f_{22} - x_2 x_4 f_{23} + x_2 x_3 f_{24} \ldots$$

$$Yf = Y\left(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3} + x_4 \frac{\partial}{\partial x_4}\right)$$

$$= -x_2 f_1 - x_1 x_2 f_{11} - x_2 x_2 f_{12} - x_2 x_3$$

	$-x_2 \frac{\partial f}{\partial x_1}$	$x_1 \frac{\partial f}{\partial x_2}$	$-x_4 \frac{\partial f}{\partial x_3}$	$x_3 \frac{\partial f}{\partial x_4}$
$x_1 \frac{\partial}{\partial x_1}$	1	B $+x_1 f_2$	2	3
$x_2 \frac{\partial}{\partial x_2}$	A $-x_2 f_1$	4	5	6
$x_3 \frac{\partial}{\partial x_3}$	7	8	9	D $+x_3 f_4$
$x_4 \frac{\partial}{\partial x_4}$	10	11	C $-x_4 f_3$	12

	$x_1 \frac{\partial f}{\partial x_1}$	$x_2 \frac{\partial f}{\partial x_2}$	$x_3 \frac{\partial f}{\partial x_3}$	$x_4 \frac{\partial f}{\partial x_4}$
$-x_2 \frac{\partial}{\partial x_1}$	1 $-x_2 f_1$	B	7	10
$x_1 \frac{\partial}{\partial x_2}$	A	4 $+x_1 f_2$	8	11
$-x_4 \frac{\partial}{\partial x_3}$	2	5	9 $-x_4 f_3$	D
$x_3 \frac{\partial}{\partial x_4}$	3	6	C	12 $+x_3 f_4$

Hence all the terms are the same and

$[X, Y] = 0.$