Geo/Top. Sp'12: 1.

(1.) M cpt, n-dim'l, show cannot be immersed in $\mathbb{R}^n$

(*) Suppose that $f: M \to \mathbb{R}^n$ is an immersion. Then $T_pf: T_pM \to T_{f(p)}\mathbb{R}^n$ is injective, i.e. $T_pf: \mathbb{R}^n \to \mathbb{R}^n$ is injective, hence surjective since dimension of domain equal to dim. of codomain, hence f has only reg. vals

Now, since M compact, im f is compact, hence closed & bdd since im $f \subseteq \mathbb{R}^n$.

Thus for $x \in \text{im} f$, $x = (x_1, \ldots, x_n)$ and $|x_i| \le b_i$ for all x_i.

Now, since im f is compact, we may choose $p_0 \in M$ s.t. $f(p_0) = (b_1, x_2, \ldots, x_n)$, i.e. p_0 such that f_1 assumes its maximum at p_0;

Since $f_1(p_0) = b_1 = \sup_{p \in M} f_1$, we know that all partial derivatives of f_1 are zero, hence the matrix $T_{p_0}f$ has a row of zeros, hence not surjective, hence p_0 is a critical point.

So we've shown f must have a critical pt, which is a contradiction to (*), hence no such immersion exists.

② $\Sigma_{1,1} = T^2 \setminus \{pt\}$

(a) $H_i(\Sigma_{1,1})$

(b) Find $H_i(\Sigma_2)$ with part (a)

(a) Apply Mayer-Vietoris

$A = \Sigma_{1,1}$, $B = D^2 \simeq pt$, $A \cup B = T^2$, $A \cap B = S^1$:

$H_2(S^1) \to H_2(\Sigma_{1,1}) \oplus H_2(pt) \to H_2(T^2) \xrightarrow{\partial} H_1(S^1) \xrightarrow{i_*} H_1(\Sigma_{1,1}) \oplus H_1(pt)$

$\to H_1(T^2) \to H_0(S^1) \to H_0(\Sigma_{1,1}) \oplus H_0(pt) \to H_0(T^2) \to 0$

$\Rightarrow 0 \to H_2(\Sigma_{1,1}) \xrightarrow{j^*} \mathbb{Z} \xrightarrow{\partial} \mathbb{Z} \xrightarrow{i_*} H_1(\Sigma_{1,1}) \xrightarrow{\delta} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\gamma} \mathbb{Z} \xrightarrow{\beta} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \to 0$

- $H_0(\Sigma_{1,1}) \cong \mathbb{Z}$ since $\Sigma_{1,1}$ is path-connected
- See that the inclusion induced homomorphism $i^*: H_p(S^1) \to H_p(\Sigma_{1,1})$ must be the zero map since S^1 is the boundary of $\Sigma_{1,1}$, hence $\operatorname{im} i_* = 0 \Rightarrow \ker i_* = \mathbb{Z} \Rightarrow \operatorname{im} \partial = \mathbb{Z} \Rightarrow \ker \partial = 0$ $\Rightarrow \operatorname{im} j^* = 0$, hence $H_2(\Sigma_{1,1}) = 0$ since j^* must be injective
- Finally, α surj. by exactness, hence $\ker \alpha = \mathbb{Z}$, hence $\operatorname{im} \beta = \mathbb{Z}$, hence $\ker \beta = 0 \Rightarrow \operatorname{im} \gamma = 0 \Rightarrow \ker \gamma = \mathbb{Z} \oplus \mathbb{Z} \Rightarrow \operatorname{im} \delta = \mathbb{Z} \oplus \mathbb{Z}$, but $\operatorname{im} i^* = 0$, hence $\ker \delta = 0$, hence $H_1(\Sigma_{1,1}) = \mathbb{Z} \oplus \mathbb{Z}$

2(b): See that $\Sigma_2 = \Sigma_{1,1} \sqcup \Sigma_{1,1} / {\text{boundaries identified}}$:

So apply Mayer-Vietoris with $A = B = \Sigma_{1,1}$, $A \cap B = S^1$, $A \cup B = \Sigma_2$:

$$0 \to H_2(S^1) \to H_2(\Sigma_{1,1}) \oplus H_2(\Sigma_{1,1}) \to H_2(\Sigma_2) \to H_1(S^1)$$
$$\to H_1(\Sigma_{1,1}) \oplus H_1(\Sigma_{1,1}) \to H_1(\Sigma_2) \to H_0(S^1) \to H_0(\Sigma_{1,1}) \oplus H_0(\Sigma_{1,1})$$
$$\to H_0(\Sigma_2) \to 0$$

Clearly Σ_2 path-connected, hence $\underline{H_0(\Sigma_2) = \mathbb{Z}}$:

$$0 \to H_2(\Sigma_2) \to \mathbb{Z} \xrightarrow{i_*} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\alpha} H_1(\Sigma_2) \xrightarrow{\beta} \mathbb{Z} \xrightarrow{\gamma} \mathbb{Z} \oplus \mathbb{Z}$$
$$\xrightarrow{\delta} \mathbb{Z} \to 0$$

- Consider map $i_* : H_1(S^1) \to H_1(\Sigma_{1,1}) \oplus H_1(\Sigma_{1,1})$; clearly both inclusions send the generator of $H_1(S^1)$ to 0 since S^1 is a boundary in $\Sigma_{1,1}$. So i_* is zero map $\Rightarrow$ $\operatorname{im} i_* = 0$ $\Rightarrow \ker i_* = \mathbb{Z} \Rightarrow \underline{H_2(\Sigma_2) = \mathbb{Z}}$ since $H_2(\Sigma_2) \to \mathbb{Z}$ injective by exactness.
- $\operatorname{im} i_* = 0 \Rightarrow \ker \alpha = 0 \Rightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} = \operatorname{im} \alpha = \ker \beta$ and $\operatorname{im} \delta = \mathbb{Z} \Rightarrow \ker \delta = \mathbb{Z} \Rightarrow \operatorname{im} \gamma = \mathbb{Z} \Rightarrow \ker \gamma = 0 \Rightarrow \operatorname{im} \beta = 0$ $\Rightarrow \ker \beta = H_1(\Sigma_2) \Rightarrow \underline{H_1(\Sigma_2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}}$

(3.) $S \subseteq \mathbb{R}^3$ ornt, embedded surface.
$\omega \in \Omega^2(S)$ s.t. $\omega_p(e_1, e_2) = 1$ for all $p \in S$, and all o.n. bases (e_1, e_2) of T_pS.
Let $n = (n_1, n_2, n_3)$ be the unit normal v.f. of S and show $\omega = n_1\, dy \wedge dz - n_2\, dx \wedge dz + n_3\, dx \wedge dy$

If (e_1, e_2) is o.n. basis for T_pS, note that $e_1 \perp n_p$ & $e_2 \perp n_p$, and furthermore, $e_1 \times e_2 = n_p = (n_1, n_2, n_3)$

Let $e_1 = (a_1, b_1, c_1)$ be an o.n. basis for $T_pS \subseteq T_p\mathbb{R}^3$
$e_2 = (a_2, b_2, c_2)$

Then $e_1 \times e_2 = \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = i(b_1c_2 - c_1b_2) - j(a_1c_2 - c_1a_2) + k(a_1b_2 - b_1a_2)$

$\Rightarrow n_1 = b_1c_2 - c_1b_2,\ -n_2 = a_1c_2 - c_1a_2,\ n_3 = a_1b_2 - b_1a_2.$

So now see that $\omega_p \in Alt^2(T_pS)$, hence
$$\omega_p = \alpha_p (dy \wedge dz)_p + \beta_p (dx \wedge dz)_p + \gamma_p (dx \wedge dy)_p$$
and then:
$$1 = \omega_p(e_1, e_2) = \alpha_p (dy \wedge dz)_p(e_1, e_2) + \beta_p (dx \wedge dz)_p(e_1, e_2) + \gamma_p (dx \wedge dy)_p(e_1, e_2)$$
$$= \alpha_p \big(dy(e_1)dz(e_2) - dy(e_2)dz(e_1)\big) + \beta_p \big(dx(e_1)dz(e_2) - dx(e_2)dz(e_1)\big)$$
$$+ \gamma_p \big(dx(e_1)dy(e_2) - dx(e_2)dy(e_1)\big)$$
$$= \alpha_p (b_1c_2 - b_2c_1) + \beta_p (a_1c_2 - a_2c_1) + \gamma_p (a_1b_2 - a_2b_1)$$
$$= \alpha_p (n_1) + \beta_p (-n_2) + \gamma_p (n_3)$$

Recall $\|n\| = \sqrt{n_1^2 + n_2^2 + n_3^2} = 1 \Rightarrow n_1^2 + n_2^2 + n_3^2 = 1$
$$\Rightarrow \underline{\alpha_p = n_1,\ \beta_p = -n_2,\ \gamma_p = n_3}$$

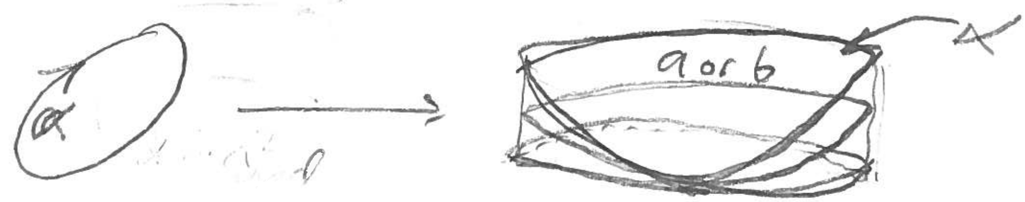
(4.) $X = M_1 \amalg M_2 / \partial M_1 \sim \partial M_2$

(a) $\pi_1(X)$?

(b) $X \simeq \Sigma_g$ some g ?

(a) Let $A = M_1$, $B = M_2$, $A \cap B = \partial M_1 \cong S^1$, $A \cup B = X$.
Now consider the inclusions $i_1 : A\cap B \hookrightarrow A$, $i_2 \, A \cap B \hookrightarrow B$
See that $\pi_1(A\cap B) \cong \pi_1(S^1) = \langle \alpha \rangle$, so:

$i_1^* : \langle \alpha \rangle \longrightarrow \pi_1(A) = \mathbb{Z}\langle a \rangle$ $\qquad$ $i_2^* : \langle \alpha \rangle \longrightarrow \pi_1(B) = \mathbb{Z}\langle b \rangle$



once around α in M_i is twice around a or b

Hence, $i_1^*(\alpha) = a^2$, $i_2^*(\alpha) = b^2$ and now apply Seifert-van Kampen

$$\pi_1(X) \cong \pi_1(A) * \pi_1(B) / \langle a^2 b^{-2} \rangle$$

$$\cong \langle a \rangle * \langle b \rangle / \langle a^2 b^{-2} \rangle \cong \underline{\langle a, b : a^2 = b^2 \rangle}$$

(b) $H_1(X) = \langle a, b : a^2 = b^2 \rangle / \langle a b a^{-1} b^{-1} \rangle$

$$= \langle a, b : a^2 b^{-2} = 1,\ a b a^{-1} b^{-1} = 1 \rangle$$

Note that $(ab^{-1})^2 = a^2 b^{-2} = b^2 b^{-2} = 1$, hence (ab^{-1}) torsion order 2; and $(ab^{-1})b = a$, so group generated by $\langle ab^{-1}, b \rangle = \langle c, b : c^2 = 1,\ cbc^{-1}b^{-1} = 1 \rangle = \mathbb{Z}_2 \oplus \mathbb{Z}$

$\rightsquigarrow$ but $H_1(M_g)$ has no torsion

(5.) Determine connected covering spaces of $\mathbb{R}P^{14} \vee \mathbb{R}P^{15}$

By the classification of connected covering spaces, the iso. classes of conn. covers are in 1-1 correspondence with the subgroups of $\pi_1(\mathbb{R}P^{14} \vee \mathbb{R}P^{15})$.

See that $\pi_1(\mathbb{R}P^{14} \vee \mathbb{R}P^{15}) = \langle a, b : a^2 = b^2 = 1 \rangle$ by Seifert-van Kampen (i.e. $\mathbb{Z}_2 * \mathbb{Z}_2$).

Recall the universal cover of $\mathbb{R}P^n$, $n > 1$, is $\pi: S^n \to \mathbb{R}P^n$.

So now:

Subgroup	Cover
$\langle e \rangle$	$S^{14} \vee S^{15}$
$\langle a \rangle$	$\mathbb{R}P^{14} \vee S^{15}$
$\langle b \rangle$	$S^{14} \vee \mathbb{R}P^{15}$
$\langle ab \rangle$	–

7:

Frobenius: ~~distribution involutive~~ $\Rightarrow$ integrable

WTS: $D = \amalg D_p = \amalg \operatorname{span}(X_p, Y_p)^{\perp}$ involutive $\quad X_p, Y_p \in T_p(\mathbb{R}^4 \setminus \{0\})$

[$D \subseteq TM$ involutive $\Leftrightarrow$ ~~If $\eta \in \Omega^1(M)$ annihilates D on open $U \subseteq M$, then $d\eta$ also annihilates D on U.~~]

If $V, W \in D$, then $[V,W] \in D \rightsquigarrow D$ involutive:

$X = (x_1, x_2, x_3, x_4) \quad X_1 = x_1,\ X_2 = x_2,\ X_3 = x_3,\ X_4 = x_4$

$Y = (-x_2, x_1, -x_4, x_3) \quad Y_1 = -x_2,\ Y_2 = x_1,\ Y_3 = -x_4,\ Y_4 = x_3$

$$[X,Y] = \sum_{j=1}^{4}\left(\sum_{i=1}^{4}\left[X^i \frac{\partial Y^j}{\partial x_i} - Y^i \frac{\partial X^j}{\partial x_i}\right]\right)\frac{\partial}{\partial x_j}$$

$$[X,Y]_j = \underbrace{\left(x_1 \frac{\partial Y^j}{\partial x_1} + x_2 \frac{\partial X^j}{\partial x_1}\right)}_{i=1} + \underbrace{\left(x_2 \frac{\partial Y^j}{\partial x_2} - x_1 \frac{\partial X^j}{\partial x_2}\right)}_{i=2} + \left(x_3 \frac{\partial Y^j}{\partial x_3} + x_4 \frac{\partial X^j}{\partial x_3}\right)$$
$$+ \left(x_4 \frac{\partial Y^j}{\partial x_4} - x_3 \frac{\partial X^j}{\partial x_4}\right)$$

$[X,Y]_1 = (x_1(0) + x_2(1)) + (x_2(-1) - x_1(0)) + (x_3(0) + x_4(0)) + 0 = x_2 - x_2 = 0$

$[X,Y]_2 = (x_1(1) + x_2(0)) + (0 - x_1(1)) + (0) + (0) = x_1 - x_1 = 0$

$[X,Y]_3 = (0) + (0) + (x_3(0) + x_4(1)) + (x_4(-1) + 0) = x_4 - x_4 = 0$

$[X,Y]_4 = (0) + (0) + (x_3(1) + 0) + (x_4(0) - x_3(1)) = x_3 - x_3 = 0$

so $[X,Y] = 0$.

(6.)

$f: M \to N$, $f_* X = Y$, ω

<u>Show</u>: $f^*(\mathcal{L}_Y \omega) = \mathcal{L}_X(f^*\omega)$ $\quad (f^*\omega = \omega \circ f_*)$

$$
\begin{aligned}
f^*(\mathcal{L}_Y \omega) &= f^*(d \circ i_Y(\omega) + i_Y \circ d\omega) \\
&= f^*(d \circ i_Y(\omega)) + f^*(i_Y \circ d\omega) \\
&= d(f^*(\omega(Y))) + \cancel{f^*(i_Y \circ d\omega)} \\
&= \underline{d(\omega(Y) \circ f)} + f^*\cancel{(i_{f_*X}(d\omega))}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_X(f^*\omega) &= d \circ i_X(f^*\omega) + i_X \circ d(f^*\omega) \\
&= d \circ (f^*\omega(X)) + \cancel{i_X \circ d(f^*\omega)} \\
&= d \circ (\omega(f_* X) \circ f) + i_X \cancel{\circ d(f^*\omega)} \\
&= \underline{d(\omega(Y) \circ f)} + i_X \cancel{\circ (f^*(d\omega))}
\end{aligned}
$$

$$
\begin{aligned}
f^*(i_Y \circ d\omega)(V) &= i_Y \circ d\omega(f_* V) \\
&= d\omega(Y, f_* V) - d\omega(f_* V, Y)
\end{aligned}
$$

$$
\begin{aligned}
i_X \circ d(f^*\omega)(V) &= i_X(f^* d\omega)(V) \\
&= f^* d\omega(X, V) - f^* d\omega(V, X) \\
&= d\omega(f_* X, f_* V) - d\omega(f_* V, f_* X) \\
&= d\omega(Y, f_* V) - d\omega(f_* V, Y)
\end{aligned}
$$