2012, Spring

Problem 1.

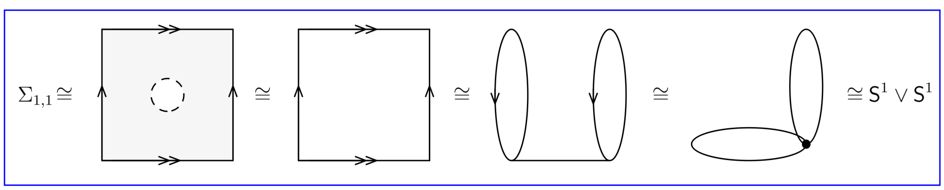
Let M be a compact n-manifold and suppose $f : M \to \mathbb{R}^n$ is an immersion; this in particular implies that $\mathsf{im}(f) \neq \varnothing$. We now have the following.

- Since M is closed and f is continuous, then $\mathsf{im}(f) \subset \mathbb{R}^n$ is closed.
- Since f is immersive, then for any $x \in M$, the map $\mathsf{d}f_x : \mathsf{T}_x M \to \mathsf{T}_{f(x)}\mathbb{R}^n$ is an injection between n-dimensional $\mathbb{R}$-vector spaces. Hence $\mathsf{d}f_x$ is an isomorphism, and so f is locally a diffeomorphism by the inverse function theorem. This implies that f is an open map, whereby $\mathsf{im}(f) \subset \mathbb{R}^n$ is open.

Thus $\mathsf{im}(f) \neq \varnothing$ is a simultaneously closed and open subspace of the connected space $\mathbb{R}^n$, whereby we must have $\mathsf{im}(f) = \mathbb{R}^n$. However this is impossible since the image of the compact space M under the continuous map f must be compact. □

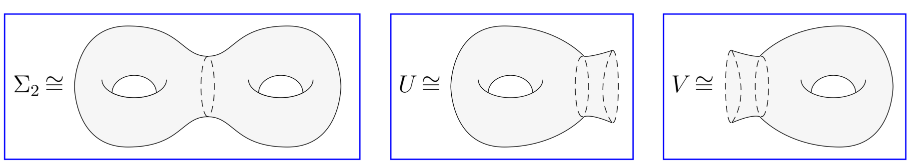
Problem 2.

(a) We stretch the missing disc inside the unit box outward until we're left with the box's (pre-identified) frame. Identifying the appropriate edges yields a wedge of two circles.



Hence $\mathsf{H}_j(\Sigma_{1,1}) \cong \mathsf{H}_j(\mathsf{S}^1 \vee \mathsf{S}^1) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, \\ 0 & \text{else.} \end{cases}$ □

(b) Decomposing Σ_2 as the union of the punctured tori U and V below, we have that $U \cong V \cong \Sigma_{1,1}$ and $U \cap V \cong \mathsf{S}^1$.



Hence by (a),

$$\mathsf{H}_j(U) \cong \mathsf{H}_j(U) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 2} & j = 1, \\ 0 & \text{else,} \end{cases} \quad \mathsf{H}_j(U \cap V) \cong \begin{cases} \mathbb{Z} & j = 0, 1, \\ 0 & \text{else.} \end{cases}$$

We already know that $\mathsf{H}_0(\Sigma_2) \cong \mathbb{Z}$ since Σ_2 is path connected. Furthermore by Mayer-Vietoris,

$$0 \longrightarrow \mathsf{H}_2(\Sigma_2) \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{(i_1,j_1)} \mathbb{Z}^{\oplus 4} \xrightarrow{k_1-\ell_1} \mathsf{H}_1(\Sigma_2) \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{(i_0,j_0)} \mathbb{Z}^{\oplus 2}$$

is exact.

- By exactness, $\ker(\partial_2) \cong 0$. Moreover, (i_1, j_1) is induced by the inclusions $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ of the boundary $U \cap V$ into either U or V, so $\mathsf{im}(i_1, j_1) \cong 0$. Hence we have $\mathsf{im}(\partial_2) \cong \ker(i_1, j_1) \cong \mathbb{Z}$, whereby $\mathsf{H}_2(\Sigma_2) \cong \mathbb{Z}$.
- By the above, $\ker(k_1 - \ell_1) \cong \mathsf{im}(i_1, j_2) \cong 0$, so $\ker(\partial_1) \cong \mathsf{im}(k_1 - \ell_1) \cong \mathbb{Z}^{\oplus 4}$. Note also that (i_0, j_0) is injective since it's induced by the inclusions i, j of path connected spaces, and so we have $\mathsf{im}(\partial_1) \cong \ker(i_0, j_0) \cong 0$. Thus $\mathsf{H}_1(\Sigma_2) \cong \mathbb{Z}^{\oplus 4}$.

Therefore $\mathsf{H}_j(\Sigma_2) \cong \begin{cases} \mathbb{Z} & j = 0, \\ \mathbb{Z}^{\oplus 4} & j = 1, \\ \mathbb{Z} & j = 2, \\ 0 & \text{else.} \end{cases}$ □

Problem 3.

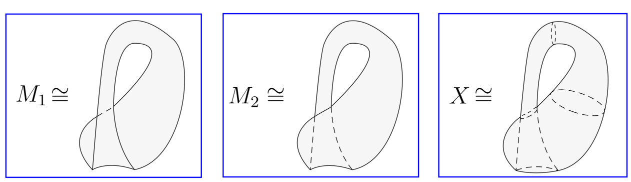
At any point $p \in S$, write $\omega_p = a_1 \mathsf{d}y \wedge \mathsf{d}z + a_2 \mathsf{d}x \wedge \mathsf{d}z + a_3 \mathsf{d}x \wedge \mathsf{d}y$ for constants $a_1, a_2, a_3 \in \mathbb{R}$, and write $e_j = (e_j^x, e_j^y, e_j^z)$ for each $j = 1, 2$. Then, using $n = (n_1, n_2, n_3) = e_1 \times e_2$, we have

$$\begin{aligned} n_1^2 + n_2^2 + n_3^2 = \|n\|^2 = 1 = \omega_p(e_1, e_2) = a_1 \mathsf{d}y \wedge \mathsf{d}z(e_1, e_2) + a_2 \mathsf{d}x \wedge \mathsf{d}z(e_1, e_2) + a_3 \mathsf{d}x \wedge \mathsf{d}y(e_1, e_2) \\ = a_1(e_1^y e_2^z - e_1^z e_2^y) + a_2(e_1^x e_2^z - e_1^z e_2^x) + a_3(e_1^x e_2^y - e_1^y e_2^x) = a_1 n_1 + a_2(-n_2) + a_3 n_3. \end{aligned}$$

Comparing the left- and right-hand sides gives $a_1 = n_1, a_2 = -n_2, a_3 = n_3$. □

Problem 4.

(a) Observe that X is the Klein bottle obtained by gluing M_1 and M_2 along their boundaries.



Firstly $M_1 \cong M_2 \cong \mathsf{S}^1$, so let x_1, x_2 be generators of $\pi_1(M_1) \cong \mathbb{Z}$ and $\pi_1(M_2) \cong \mathbb{Z}$, respectively. Moreover $M_1 \cap M_2 \cong \partial M_1 \cong \mathsf{S}^1$, so for each $j = 1, 2$, the inclusion $\iota_j : M_1 \cap M_2 = \partial M_j \hookrightarrow M_j$ winds the loop $1 \in \pi_1(M_1 \cap M_2) \cong \pi_1(\mathsf{S}^1) \cong \mathbb{Z}$ once over the "front" of M_j and once over the "back," so that $\iota_j(1) = x_j^2$. Then by van Kampen,

$$\pi_1(X) \cong \pi_1(M_1) *_{\pi_1(M_1 \cap M_2)} \pi_1(M_2) \cong \frac{\langle x_1, x_2 \rangle}{\langle \iota_1(1)\iota_2(1)^{-1} \rangle} \cong \langle x_1, x_2 \mid x_1^2 = x_2^2 = 1 \rangle.$$

□

(b) **No**. The Klein bottle is unorientable. □

Problem 5.

Equivalence classes of connected covers of $\mathbb{R}\mathsf{P}^{14} \vee \mathbb{R}\mathsf{P}^{15}$ are in bijection with the subgroups of

$$\pi_1(\mathbb{R}\mathsf{P}^{14} \vee \mathbb{R}\mathsf{P}^{15}) \cong \pi_1(\mathbb{R}\mathsf{P}^{14}) * \pi_1(\mathbb{R}\mathsf{P}^{15}) \cong \mathbb{Z}_2 * \mathbb{Z}_2 \cong \langle x, y \mid x^2 = y^2 = 1 \rangle.$$

The identity subgroup corresponds to the universal cover $\mathsf{S}^{14} \vee \mathsf{S}^{15}$; the entire group corresponds to the trivial cover $\mathbb{R}\mathsf{P}^{14} \vee \mathbb{R}\mathsf{P}^{15}$; the subgroups generated by x and y correspond to the covers $\mathbb{R}\mathsf{P}^{14} \vee \mathsf{S}^{15}$ and $\mathsf{S}^{14} \vee \mathbb{R}\mathsf{P}^{15}$, respectively. □

Problem 6.

By Cartan,

$$f^*(\mathcal{L}_Y \omega) = f^*(\iota_Y \mathsf{d}\omega) + f^*(\mathsf{d}\iota_Y \omega), \quad \mathcal{L}_X(f^*\omega) = \iota_X(\mathsf{d}f^*\omega) + \mathsf{d}\iota_X(f^*\omega).$$

We show that the right-hand sides are equal by showing equality of the corresponding summands. We have for the second summand

$$f^*(\mathsf{d}\iota_Y\omega) = \mathsf{d}f^*(\iota_Y\omega) = \mathsf{d}f^*(\omega(Y)) = \mathsf{d}(\omega(Y) \circ f) = \mathsf{d}(\omega(f_*(X)) \circ f) = \mathsf{d}((f^*\omega)(X)) = \mathsf{d}\iota_X(f^*\omega).$$

Next, for any $x \in M$ and $v \in \mathsf{T}_x M$, we have for the first summand

$$\begin{aligned}(f^*(\iota_Y \mathsf{d}\omega))_x(v) &= (\iota_Y \mathsf{d}\omega)_{f(x)}(f_* v) = \mathsf{d}\omega_{f(x)}(Y(f(x)), f_* v) = \mathsf{d}\omega_{f(x)}(f_*(X(x)), f_* v) \\ &= (f^*\mathsf{d}\omega)_x(X(x), v) = (\iota_X(f^*\mathsf{d}\omega))_x(v) = (\iota_X(\mathsf{d}f^*\omega))_x(v),\end{aligned}$$

and so $f^*(\iota_Y \mathsf{d}\omega) = \iota_X(\mathsf{d}f^*\omega)$. □

Problem 7.

Remark. It's indeed the case that $[X, Y] = 0$, but this only tells us by Frobenius that the rank-2 distribution defined by X and Y (not the one orthogonal to X and Y) is integrable.

No. Denote by $\mathscr{D}$ the rank-2 distribution orthogonal to X and Y, and begin by taking some arbitrary vector field $V = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v_3 \frac{\partial}{\partial x_3} + v_4 \frac{\partial}{\partial x_4} \in \mathscr{D}$. Then

$$\begin{aligned}\langle V, X \rangle = 0 &\implies v_1 x_1 + v_2 x_2 + v_3 x_3 + v_4 x_4 = 0, \\ \langle V, Y \rangle = 0 &\implies -v_1 x_2 + v_2 x_1 - v_3 x_4 + v_4 x_3 = 0,\end{aligned}$$

so we have the matrix equation

$$\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} x_3 & x_4 \\ -x_4 & x_3 \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = 0.$$

By assumption, $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \setminus 0$, so w.l.o.g. $x_1 \neq 0$. Then this equation gives

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}^{-1} \begin{pmatrix} x_3 & x_4 \\ -x_4 & x_3 \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = -\frac{1}{x_1^2 + x_2^2} \begin{pmatrix} x_1 x_3 + x_2 x_4 & x_1 x_4 - x_2 x_3 \\ x_2 x_3 - x_1 x_4 & x_1 x_3 + x_2 x_4 \end{pmatrix} \begin{pmatrix} v_3 \\ v_4 \end{pmatrix}.$$

Now, we may freely choose $(v_3, v_4) \in \mathbb{R}^2$ and this equation determines $(v_1, v_2) \in \mathbb{R}^2$ such that the resulting vector field V belongs to $\mathscr{D}$. Setting $(v_3, v_4) = -(x_1^2 + x_2^2, 0)$ and $(v_3', v_4') = -(0, x_1^2 + x_2^2)$, respectively, we obtain two sets of coefficients

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} x_1 x_3 + x_2 x_4 \\ x_2 x_3 - x_1 x_4 \\ -x_1^2 - x_2^2 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_1' \\ v_2' \\ v_3' \\ v_4' \end{pmatrix} = \begin{pmatrix} x_1 x_4 - x_2 x_3 \\ x_1 x_3 + x_2 x_4 \\ 0 \\ -x_1^2 - x_2^2 \end{pmatrix}$$

which yield, respectively, two vector fields $V, V' \in \mathscr{D}$. But now, $[V, V'] = -2(x_1^2 + x_2^2)Y$, whereby $[V, V'] \notin \mathscr{D}$ since $\langle -2(x_1^2 + x_2^2)Y, Y \rangle \neq 0$ for $x_1 \neq 0$. Thus $\mathscr{D}$ is nonintegrable by Frobenius. □